Semiring Neighbours

An Algebraic Embedding and Extension of Neighbourhood Logic

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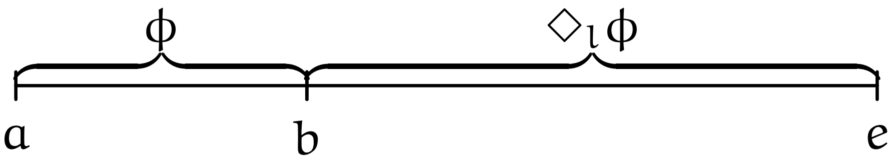
Aim

- embed and extend NL
- get additional results for NL
- generalise the existing results for NL
- adopt the results to other areas, like graphs and hybrid systems

1 About Neighbourhood Logic

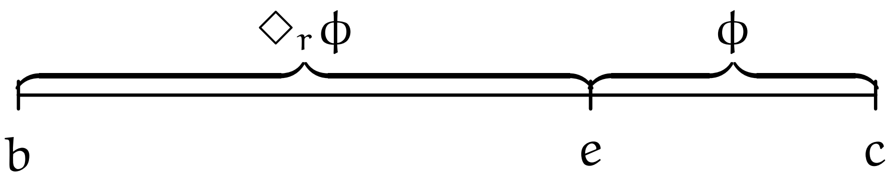
- purpose: reasoning about time intervals
- in particular, about neighbouring intervals
- chop-based interval temporal logics, like ITL and IL, cannot express all desired properties
- first-order interval logic
- introduced by Zhou and Hansen in 1996
  expanded by Zhou and Roy
- main idea: *left* and *right neighbourhoods* as primitive intervals

left neighbourhood: $\Diamond_l \phi$



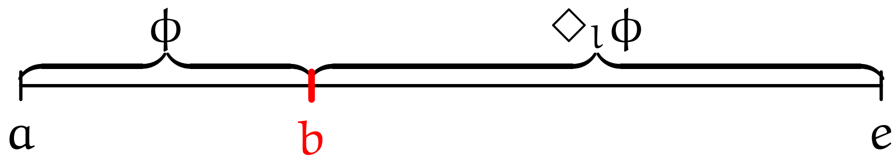
$\Diamond_l \phi$ holds on $[b, e]$ iff there exists $a \leq b$ such that ϕ holds on $[a, b]$

right neighbourhood: $\Diamond_r \phi$



$\Diamond_r \phi$ holds on $[b, e]$ iff there exists $c \geq e$ such that ϕ holds on $[e, c]$

- *expanding* modalities
- neighbours only depends on *contact points*



- characterise those as action points of sequential composition

2 Some short definitions

Definition 2.1 *idempotent semiring* $(S, +, \cdot, 0, 1)$:

- $(S, +, 0)$ commutative monoid (+ is choice)
- $(S, \cdot, 1)$ monoid (· is composition)
- multiplication is distributive

  $(a + b) \cdot c \; = \; a \cdot c + b \cdot c \qquad a \cdot (b + c) \; = \; a \cdot b + a \cdot c$
- 0 is annihilator

  $0 \cdot a = 0 = 0 \cdot a$
- $a + a = a$

natural order: $a \leq b \; \Leftrightarrow \; a + b = b$

Definition 2.2 *test semiring* $(S, \mathsf{test}(S))$:

- S idempotent semiring
- $\mathsf{test}(S) \subseteq [0, 1]$ Boolean algebra (abstract assertions)

Definition 2.3 *domain semiring* $(S, \ulcorner)$:

- S test semiring
- $\ulcorner : S \to \mathsf{test}(S)$

$$a \leq \ulcorner a \cdot a \qquad \ulcorner(p \cdot a) \leq p$$

analogously: *codomain semiring*

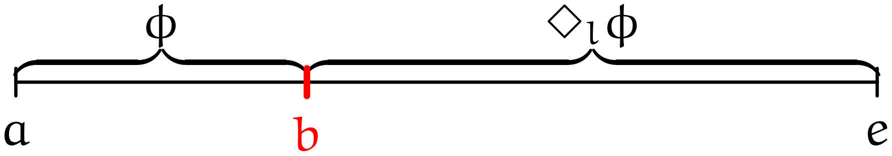
work on semirings with domain *and* codomain

e.g.: [DesharnaisMöllerStruth04]

2.1 Examples

- algebra of intervals
  - elements: sets of intervals
  - $\ulcorner$: starting points, i.e., $\ulcorner I = \{[a,a] : [a,x] \in I\}$
  - $\urcorner$: ending points
- algebra of binary relations under relational composition
  - $\ulcorner$: domain of a relation, i.e., $\ulcorner R = \{(a,a) : (a,x) \in R\}$
  - $\urcorner$: range of a relation
- path algebra under path fusion
  - elements: sets of paths in a given graph
  - $\ulcorner$: starting nodes
  - $\urcorner$: ending points
- ...

3 Embedding of NL



- b is starting point of $[b, e]$
- b is ending point of $[a, b]$

Definition 3.1

x is a *left neighbour* of y (or for short: $x \leq \Diamond_l y$) iff $x^{\urcorner} \leq {}^{\ulcorner}y$

x is a *right neighbour* of y (or for short: $x \leq \Diamond_r y$) iff ${}^{\ulcorner}x \leq y^{\urcorner}$

Let $[\![\phi]\!]$ be the set of all intervals where ϕ holds.

Lemma 3.2

$$\Diamond_r \phi \text{ holds on } x \Leftrightarrow x \leq \textcircled{n}_l [\![\phi]\!] \Leftrightarrow x^{\rceil} \leq {}^{\ulcorner}([\![\phi]\!])$$

$$\Diamond_l \phi \text{ holds on } x \Leftrightarrow x \leq \textcircled{n}_r [\![\phi]\!] \Leftrightarrow {}^{\ulcorner}x \leq ([\![\phi]\!])^{\rceil}$$

- thus, NL is embedded into semirings
- NL can be adopted to other interpretations, like graphs

more neighbours (briefly)

- perfect neighbours (box operators)

$$\Box_l \phi \stackrel{\text{def}}{=} \neg \Diamond_l \neg \phi \text{ holds on } x \qquad \Leftrightarrow \qquad (\llbracket \neg\phi \rrbracket)^{\urcorner} \cdot {}^{\ulcorner}x \leq 0$$

$$\Box_r \phi \stackrel{\text{def}}{=} \neg \Diamond_r \neg \phi \text{ holds on } x \qquad \Leftrightarrow \qquad x^{\urcorner} \cdot {}^{\ulcorner}(\llbracket \neg\phi \rrbracket) \leq 0$$

$$x \leq \boxed{\sqcap}_r y \stackrel{\text{def}}{\Leftrightarrow} \overline{y}^{\urcorner} \cdot {}^{\ulcorner}x \leq 0 \qquad\qquad x \leq \boxed{\sqcap}_l y \stackrel{\text{def}}{\Leftrightarrow} x^{\urcorner} \cdot {}^{\ulcorner}\overline{y} \leq 0$$

- combinations $\Diamond_l \Diamond_r \phi$, $\Diamond_r \Diamond_l \phi$

$${}^{\ulcorner}x \leq {}^{\ulcorner}y \qquad\qquad x^{\urcorner} \leq y^{\urcorner}$$

- boxes of combinations

$${}^{\ulcorner}x \cdot {}^{\ulcorner}\overline{y} \leq 0 \qquad\qquad \overline{y}^{\urcorner} \cdot x^{\urcorner} \leq 0$$

4 Results

- underlying theory
  - de Morgan dualities
  - Galois connections, like

    $\Diamond\!\!\!\!\!{\scriptstyle n}\;\;_l x \leq y \Leftrightarrow x \leq \boxed{n}_r y \qquad \Diamond\!\!\!\!\!{\scriptstyle n}\;\;_r x \leq y \Leftrightarrow x \leq \boxed{n}_l y$
- simplifying NL
  - at least two axioms of NL can be dropped
  - additional box operators
  - most of the properties of [ZhouHansen] follow from the Galois connections
  - there are explicit expressions for neighbours, e.g. $\Diamond\!\!\!\!\!{\scriptstyle n}\;\;_l y = \top \cdot \ulcorner y$
- almost all results of NL can be lifted to semirings

5 Interpretation in other semirings

- algebra of binary relations:

  neighbours: permeability

  $$\Diamond\!\!\!\!\!{\scriptstyle n}\;_r R = \{(x,y) \mid \exists\, w : (w,x) \in R,\ y \in M\}$$

  perfect neighbours: full permeability

  $$\Box\!\!\!\!\!{\scriptstyle n}\;_r R = \{(x,y) \mid \forall\, w : (w,x) \in R,\ y \in M\}$$

  the relation you can *only* reach through R

- path algebra:

  neighbours: reachability

  perfect neighbours: exclusive reachability

Interpretation in hybrid systems

- a *trajectory* is a pair (i, f) of an interval i and a function $f : i \to V$
- only finite trajectories !
- $(\wp(\mathrm{TRA}), \cup, \circ, \emptyset, \mathbb{1})$ forms a semiring
- allows statements about neighbours (similar to the original idea)
- interpretation of neighbours: again some kind of reachability

[HöfnerMöller05]

6 Conclusion and Outlook

done

- embeded NL into semirings
- expanded NL by additional operators and theorems
- simplified NL (some axioms are theorems in our approach)

to do

- expand neighbours to Lazy semirings [Möller04]
- include infinite elements
- get interpretation of hybrid systems with trajectories of finite and infinity length [HöfnerMöller05]