Non-Termination in Idempotent Semirings

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Abstract. We study and compare two notions of non-termination on idempotent semirings: infinite iteration and divergence. We determine them in various models and develop conditions for their coincidence. It turns out that divergence yields a simple and natural way of modelling infinite behaviour, whereas infinite iteration shows some anomalies.

1 Introduction

Idempotent semirings and Kleene algebras have recently been established as foundational structures in computer science. Initially conceived as algebras of regular expressions, they now find widespread applications ranging from program analysis and semantics to combinatorial optimisation and concurrency control.

Kleene algebras provide operations for modelling actions of programs or transition systems under non-deterministic choice, sequential composition and finite iteration. They have been extended by omega operations for infinite iteration [2, 16], by domain and modal operators [4, 12] and by operators for program divergence [3]. The resulting formalisms bear strong similarities with propositional dynamic logics, but have a much richer model class that comprises relations, paths, languages, traces, automata and formal power series.

Among the most fundamental analysis tasks for programs and reactive systems are termination and non-termination. In a companion paper [3], different algebraic notions of termination based on modal semirings have been introduced and compared. The most important ones are the omega operator for infinite iteration [2] and the divergence operator which models that part of a state space from which infinite behaviour may arise. Although, intuitively, absence of divergence and that of infinite iteration should be the same concept, it was found that they differ on some very natural models, including languages.

Here, we extend this investigation to the realm of non-termination. Our results further confirm the anomalies of omega. They also suggest that the divergence semirings proposed in [3] are powerful tools that capture terminating and non-terminating behaviour on various standard models of programs and reactive systems; they provide the right level of abstraction for analysing them in simple and concise ways. Our main contributions are as follows.

- We systematically compare infinite iteration and divergence in concrete models, namely finite examples, relations, traces, languages and paths. The concepts coincide in relation semirings, but differ on all other models considered.

- We also study abstract *taming conditions* for omega that imply coincidence with divergence. We find a rather heterogenous situation: Omega is tame on relation semirings. It is also tame on language semirings, but violates the taming condition. Therefore, the taming condition is only sufficient, but not necessary. In particular, omega is not tame on trace and path semirings.

The approach uses general results about fixed points for characterising and computing iterations in concrete models. Standard techniques from universal algebra relate the infinite models by Galois connections and homomorphisms.

All proofs at the level of Kleene algebras have been done by the automated theorem prover Prover9 [10]. They are documented at a website [7] and can easily be reproduced using the template in Appendix A. Proofs that use properties of particular models are given in Appendix B.

2 Idempotent Semirings and Omega Algebras

Our algebraic analysis of non-termination is based on idempotent semirings.

A *semiring* is a structure $(S, +, \cdot, 0, 1)$ such that $(S, +, 0)$ is a commutative monoid, $(S, \cdot, 1)$ is a monoid, multiplication distributes over addition and 0 is a left and right zero of multiplication. A semiring S is *idempotent* (an *i-semiring*) if $(S, +)$ is a semilattice with $x + y = \sup(x, y)$. (See the Prover9 input files in Appendix A for the axioms.)

Idempotent semirings are useful for modelling actions, programs or state transitions under non-deterministic choice and sequential composition. We usually omit the multiplication symbol. The semilattice-order $\leq$ on S has 0 as its least element; addition and multiplication are isotone with respect to it.

Tests of a program or sets of states of a transition system can also be modelled in this setting. A *test* in an i-semiring S is an element of a Boolean subalgebra $\mathsf{test}(S) \subseteq S$ (the *test algebra* of S) such that $\mathsf{test}(S)$ is bounded by 0 and 1 and multiplication coincides with lattice meet. We will write $a, b, c \ldots$ for arbitrary semiring elements and $p, q, r, \ldots$ for tests. We will freely use the standard laws of Boolean algebras on tests.

Iteration can be modelled on i-semirings by adding two operations.

A *Kleene algebra* [9] is an i-semiring S extended by an operation $^* : S \to S$ that satisfies the *star unfold* and *star induction* axioms

$$1 + aa^* \leq a^*, \quad 1 + a^*a \leq a^*, \quad b + ac \leq c \Rightarrow a^*b \leq c, \quad b + ca \leq c \Rightarrow ba^* \leq c.$$

An *omega algebra* [2] is a Kleene algebra S extended by an operation $^\omega : S \to S$ that satisfies the *omega unfold* and the *omega co-induction* axiom

$$a^\omega \leq aa^\omega, \qquad c \leq b + ac \Rightarrow c \leq a^\omega + a^*b.$$

a^*b and $a^\omega + a^*b$ are the least and the greatest fixed point of $\lambda x.b + ax$. The least fixed point of $\lambda x.1 + ax$ is a^* and a^ω is the greatest fixed point of $\lambda x.ax$.

The star and the omega operator are intended to model finite and infinite iteration on i-semirings; Kleene algebras and omega algebras are intended as algebras of regular and ω-regular events. A particular strength is that they allow first-order equational reasoning and therefore automated deduction [8]. Since i-semirings are an equational class, they are, by Birkhoff's HSP-theorem, closed under subalgebras, direct products and homomorphic images. Furthermore, since Kleene algebras and omega algebras are universal Horn classes, they are, by further standard results from universal algebra, closed under subalgebras and direct products, but not in general under homomorphic images. We will use these facts for constructing new algebras from given ones. Finite equational axiomatisations of algebras of regular events are ruled out since Kleene algebras are (sound and) complete for the equational theory of regular expressions, but there is no finite equational axiomatisation for this theory [9].

Consequently, all regular identities hold in Kleene algebras and we will freely use them. Examples are $0^* = 1 = 1^*$, $1 \leq a^*$, $aa^* \leq a^*$, $a^*a^* = a^*$, $a \leq a^*$, $a^*a = aa^*$ and $1 + aa^* = a^* = 1 + a^*a$. Furthermore the star is isotone.

It has also been shown that ω-regular identities such as $0^\omega = 0$, $a \leq 1^\omega$, $a^\omega = a^\omega 1^\omega$, $a^\omega = aa^\omega$, $a^\omega b \leq a^\omega$, $a^*a^\omega = a^\omega$ and $(a+b)^\omega = (a^*b)^\omega + (a^*b)^*a^\omega$ hold in omega algebras and that omega is isotone. Automated proofs of all these identities can be found at our website [7]. However, omega algebras are not complete for the equational theory of ω-regular expressions: Products of the form ab exist in ω-regular languages only if a represents a set of finite words whereas no such restriction is imposed on omega algebra terms. Moreover, every omega algebra has a greatest element $\top = 1^\omega$, and the following property holds [7].

$$(a + p)^\omega = a^\omega + a^*p\top. \tag{1}$$

3 Iterating Star and Omega

We will consider several important models in which a^* and a^ω do exist and in which a^* can be determined by fixed point iteration via the Knaster-Tarski theorem, whereas a^ω could only exist under additional assumptions that do not generally hold in our context. We will now set up the general framework.

One way to guarantee the existence of a^* and a^ω is to assume a *complete* i-semiring, i.e., an i-semiring with a complete semilattice reduct. Since every complete semilattice is also a complete lattice, a^* and a^ω exist and a^* can be approximated by $\sup(a^i : i \in \mathbb{N}) \leq a^*$ along the lines of Knaster-Tarski, where sup denotes the supremum operator. An iterative computation of a^*b presumes the additional infinite distributivity law

$$\sup(a^i : i \in \mathbb{N})b = \sup(a^ib : i \in \mathbb{N})$$

and similarly for ba^*. Such infinite laws always hold when the lattice reduct of the i-semiring is complete, Boolean, and meet coincides with multiplication. In particular, all *finite* i-semirings and all i-semirings defined on powersets with multiplication defined via pointwise extension are complete and the infinite distributivity laws hold. In all these cases, a^* can be iteratively determined as

$$a^* = \sup(a^i : i \in \mathbb{N})$$

and a^* is the reflexive transitive closure of a. Alternatively, the connection of a^* and iteration via suprema could be enforced by continuity [9].

It would be tempting to conjecture a dual iteration for a^ω. This would, however, presuppose distributivity of multiplication over arbitrary infima, which is not the case (cf. [13] for a counterexample). In general, we can only expect that

$$a^\omega \leq \inf(a^i\top : i \in \mathbb{N}).$$

An exception is the finite case, where every isotone function is also co-continuous. In this particular case, therefore $a^\omega = \inf(a^i\top : i \in \mathbb{N})$, i.e., a^ω can be iterated from the greatest element of a finite omega-algebra.

We will now illustrate the computation of star and omega in a simple finite relational example. This example will also allow us to motivate some concepts and questions that are treated in later sections.

Example 3.1. Consider the binary relation a in the first graph of Figure 1.

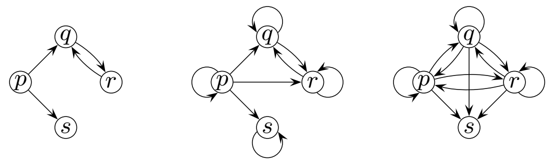

Fig. 1. The relations a, a^* and a^ω.

Iterating $a^* = \sup(a^i : i \in \mathbb{N})$ yields the second graph of Figure 1. a^* represents the finite a-paths by collecting their input and output points: $(x, y) \in a^*$ iff there is a finite a-path from x to y.

Analogously one might expect that a^ω represents infinite a-paths in the sense that $(x, y) \in a^\omega$ iff x and y lie on an infinite a-path. However, iterating $a^\omega = \inf(a^i\top : i \in \mathbb{N})$ yields the right-most graph of Figure 1. It shows that $(q, p) \in a^\omega$ although there is no a-path from q to p, neither finite nor infinite.

So what does a^ω represent? Let ∇a model those nodes from which a diverges, i.e., from which an infinite a-path emanates. Then Example 3.1 shows that elements in ∇a are linked by a^ω to any other node; elements outside of ∇a are not in the domain of a^ω. Interpreting a^ω generally as *anything for states on which a diverges* would be consistent with the demonic semantics of total program correctness; its interpretation of *nothing for states on which a diverges* models partial correctness. This suggests to further investigate the properties

$$(\nabla a)\top = a^\omega \quad \text{and} \quad \nabla a = \mathsf{dom}(a^\omega).$$

These two identities do not only hold in Example 3.1; they will be of central interest in this paper. To study them further, we will now introduce some important models of i-semirings and then formalise divergence in this setting.

4 Omega on Finite Idempotent Semirings

We have explicitly computed the stars and omegas for some small finite models using the model generator Mace4 [10]. We will further analyse these models in Section 9 and use them as counterexamples in Section 10.

Example 4.1. The two-element Boolean algebra is an i-semiring and an omega algebra with $0^* = 1^* = 1^\omega = 1$ and $0^\omega = 0$. It is the only two-element omega algebra and denoted by A_2.

Example 4.2. There are three three-element i-semirings. Their elements are from $\{0, a, 1\}$. Only a is free in the defining tables. Stars and omegas are fixed by $0^* = 1^* = 1$, $0^\omega = 0$ and $1^\omega = \top$ (the greatest element) except for a.

(a) In A_3^1, addition is defined by $0 < 1 < a$, moreover, $aa = a^* = a^\omega = a$.
(b) In A_3^2, $0 < a < 1$, $aa = a^\omega = 0$ and $a^* = 1$.
(c) In A_3^3, $0 < a < 1$, $aa = a^\omega = a$ and $a^* = 1$.

5 Trace, Path and Language Semirings

We now present some of the most interesting models of i-semirings: traces, paths and languages. These are well-known; we formally introduce them only since we will study divergence and omega on these models in later sections.

As usual, a *word* over a set Σ is a mapping $[0..n] \to \Sigma$. The empty word is denoted by ε and *concatenation* of words σ_0 and σ_1 by $\sigma_0.\sigma_1$. We write $\mathsf{first}(\sigma)$ for the first element of a word σ and $\mathsf{last}(\sigma)$ for its last element. We write $|\sigma|$ for the length of σ. The set of all words over Σ is denoted by Σ^*.

A (finite) *trace* over the sets P and A is either ε or a word σ such that $\mathsf{first}(\sigma)$, $\mathsf{last}(\sigma) \in P$ and in which elements from P and A alternate. $\tau_0, \tau_1, \ldots$ will denote traces. For $s \in P$ the product of traces τ_0 and τ_1 is the trace

$$\tau_0 \cdot \tau_1 = \begin{cases} \sigma_0.s.\sigma_1 & \text{if } \tau_0 = \sigma_0.s \text{ and } \tau_1 = s.\sigma_1, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Intuitively, $\tau_0 \cdot \tau_1$ glues two traces together when the last state of τ_0 and the first state of τ_1 are equal. The set of all traces over P and A is denoted by $(P, A)^*$, where P is the set of *states* and A the set of *actions*.

Lemma 5.1. *The power-set algebra $2^{(P,A)^*}$ with addition defined by set union, multiplication by $S \cdot T = \{\tau_0 \cdot \tau_1 : \tau_0 \in S, \tau_1 \in T$ and $\tau_0 \cdot \tau_1$ defined$\}$, and with $\emptyset$ and P as neutral elements is an i-semiring.*

We call this i-semiring the *full trace semiring* over P and A. By definition, $S \cdot T = \emptyset$ if all products between traces in S and traces in T are undefined.

Every subalgebra of the full trace semiring is, by the HSP-theorem, again an i-semiring (constants such as 0, 1 and $\top$ are fixed by subalgebra constructions). We will henceforth consider only complete subalgebras of full trace semirings and call them *trace semirings*. Every non-complete subalgebra of the full trace semiring can of course uniquely be closed to a complete subalgebra.

As we will see, forgetting parts of the structure is quite useful. First we want to forget all actions of traces. Consider the *projection* $\phi_P : (P, A)^* \to P^*$ which is defined, for all $s \in P$ and $\alpha \in A$ by

$$\phi_P(\varepsilon) = \varepsilon, \qquad \phi_P(s.\sigma) = s.\phi_P(\sigma), \qquad \phi_P(\alpha.\sigma) = \phi_P(\sigma).$$

ϕ_P is a mapping between traces and words over P which we call *paths*. Moreover it can be seen as the homomorphic extension of the function $\phi(\varepsilon) = \phi(\alpha) = \varepsilon$ and $\phi(s) = s$ with respect to concatenation. A product on paths can be defined as for traces. Again, $\pi_0 \cdot \pi_1$ glues two paths π_0 and π_1 together when the last state of π_0 and the first state of π_1 are equal.

The mapping ϕ_P can be extended to a set-valued mapping $\phi_P : 2^{(P,A)^*} \to 2^{P^*}$ by taking the image, i.e., $\phi_P(T) = \{\phi_P(\tau) : \tau \in T\}$. Now, ϕ_P sends sets of traces to sets of paths.

The information about actions can be introduced to paths by *fibration*, which can be defined in terms of the relational inverse $\phi_P^{-1} : P^* \to 2^{(P,A)^*}$ of ϕ_P. Intuitively, it fills the spaces between states in a path with all possible actions and therefore maps a single path to a set of traces. The mapping ϕ_P^{-1} can as well be lifted to the set-valued mapping $\phi_P^\sharp(Q) = \sup(\phi_P^{-1}(\pi) : \pi \in Q)$, where $Q \in 2^{P^*}$ is a set of paths.

Lemma 5.2. *ϕ_P and $\phi_P^\sharp$ are adjoints of a Galois connection, i.e., for $a \in 2^{(P,A)^*}$ and $b \in 2^{P^*}$ we have*

$$\phi_P(a) \le b \Leftrightarrow a \le \phi_P^\sharp(b).$$

The proof of this fact is standard. Galois connections are interesting because they give theorems for free. In particular, ϕ_P commutes with all existing suprema and $\phi_P^\sharp$ commutes with all existing infima. Also, ϕ_P is isotone and $\phi_P^\sharp$ is antitone. Both mappings are related by the cancellation laws $\phi_P \circ \phi_P^\sharp \le id_{2^{P^*}}$ and $id_{2^{(P,A)^*}} \le \phi_P^\sharp \circ \phi_P$. Finally, the mappings are pseudo-inverses, that is, $\phi_P \circ \phi_P^\sharp \circ \phi_P = \phi_P$ and $\phi_P^\sharp \circ \phi_P \circ \phi_P^\sharp = \phi_P^\sharp$.

Lemma 5.3. *The mappings ϕ_P are homomorphisms.*

By the HSP-theorem the set-valued homomorphism induces path semirings from trace semirings.

Lemma 5.4. *The power-set algebra 2^{P^*} is an i-semiring.*

We call this i-semiring the *full path semiring* over P. It is the homomorphic image of a full trace semiring. Again, by the HSP-theorem, all subalgebras of full path semirings are i-semirings; complete subalgebras are called *path semirings*.

Lemma 5.5. *Every identity that holds in all trace semirings holds in all path semirings.*

Moreover, the class of trace semirings contains isomorphic copies of all path semirings. This can be seen as follows.

Consider the congruence $\sim_P$ on a trace semiring over P and A that is induced by the homomorphism ϕ_P. The associated equivalence class $[T]_P$ contains all those sets of traces that differ in actions, but not in paths. From each equivalence class we can choose as canonical representative a set of traces all of which are built from one single action. Each of these representatives is of course equivalent to a set of paths and therefore an element of a path semiring. Conversely, every element of a path semiring can be expanded to an element of some trace semiring by filling in the same action between all states.

The following lemma can be proved using techniques from universal algebra.

Lemma 5.6. *Let S be the full trace semiring over P and A. The quotient algebra $S/{\sim_P}$ is isomorphic to each full trace semiring over P and $\{a\}$ with $a \in A$ and to the full path semiring over P:*

$$S/{\sim_P} \;\cong\; 2^{(P,\{a\})^*} \;\cong\; 2^{P^*}.$$

In particular, the mappings ϕ_P and $\phi_P^\sharp$ are isomorphisms between the full trace semiring $2^{(P,\{a\})^*}$ and the full path semiring 2^{P^*}. In that case, $\phi_P^{-1} = \phi_P^\sharp$.

Lemma 5.6 is not only limited to full trace and path semirings. It immediately extends to trace and path semirings, since the operations of forming subalgebras and of taking homomorphic images always commute. In particular, each path semiring is isomorphic to some trace semiring with a single action. This isomorphic embedding of path semirings into the class of trace semirings implies the following proposition.

Proposition 5.7. *Every first-order property that holds in all trace semirings holds in all path semirings.*

In particular, Horn clauses that hold in all trace semirings are also valid in the setting of paths.

A similar mapping and Galois connection for languages can be defined by forgetting states, but it does not extend to a homomorphism: forgetting states before or after products yields different results. Nevertheless, the class of trace semirings contains again elements over one single state. These are isomorphic to (complete) *language semirings*, which are algebras of formal languages. Conversely, every language semiring can be induced by this isomorphism.

Proposition 5.8. *Every first-order property that holds in all trace semirings holds in all language semirings.*

6 Relation Semirings

Now we forget entire paths between the first and the last state of a trace. We therefore consider the mapping $\phi_R : (P, A)^* \to P \times P$ defined by

$$\phi_R(\tau) = \begin{cases} (\mathsf{first}(\tau), \mathsf{last}(\tau)) & \text{if } \tau \neq \varepsilon, \\ \text{undefined} & \text{if } \tau = \varepsilon. \end{cases}$$

It sends trace products to (standard) relational products on pairs. As before, ϕ_R can be extended to a set-valued mapping $\phi_R : 2^{(P,A)^*} \to 2^{P \times P}$ by taking the image, i.e., $\phi_R(T) = \{\phi_R(\tau) : \tau \in T\}$. Now, ϕ_R sends sets of traces to *relations*. Information about the traces between starting and ending state can be introduced to pairs of states by the fibration $\phi_R^{-1} : P \times P \to 2^{(P,A)^*}$ of ϕ_R. Intuitively, it replaces a pair of states by all possible traces between them. It can again be lifted to the set-valued mapping $\phi_R^\sharp(R) = \sup(\phi_R^{-1}(r) : r \in R)$, for any relation $R \in 2^{P \times P}$.

Lemma 6.1. *ϕ_R and $\phi_R^\sharp$ are adjoints of a Galois connection.*

The standard properties hold again.

Lemma 6.2. *The mappings ϕ_R are homomorphisms.*

By the HSP-theorem, the set-valued homomorphism induces relation semirings from trace semirings.

Lemma 6.3. *The power-set algebra $2^{P \times P}$ is an i-semiring.*

We call this i-semiring the *full relation semiring* over P. It is the homomorphic image of a full trace semiring. Again, all subalgebras of full relation semirings are i-semirings; complete subalgebras are called *relation semirings.*

Proposition 6.4. *Every identity that holds in all trace semirings holds in all relation semirings.*

Similar to $\sim_P$ we can define $\sim_R$ induced by ϕ_R. But in that case, multiplication is not well-defined in general and the quotient structures induced are not semirings.

Lemma 6.5. *There is no trace semiring over P and A that is isomorphic to the full relation semiring over a finite set Q with $|Q| > 1$.*

A homomorphism that sends path semirings to relation semirings can be built in the same way as ϕ_R and $\phi_R^\sharp$, but using paths instead of traces as an input. The homomorphism $\chi : 2^{A^*} \to 2^{A^* \times A^*}$ that sends language semirings to relation semirings uses a standard construction (cf. [14]). It is defined, for all $L \subseteq A^*$ by $\tilde{\chi}(L) = \{(v, v.w) : v \in A^* \text{ and } w \in L\}$.

Lemma 6.6. *Every identity that holds in all path or language semirings holds in all relation semirings.*

It is important to distinguish between relation semirings and relational structures under addition and multiplication in general.

We will often need to consider trace semirings and relation semirings separately, whereas language and path semirings are subsumed.

7 Omega on Trace, Language and Path Semirings

Let us consider star and omega in (infinite) trace, path and language semirings. We will relate the results obtained with divergence in Section 9. We will also study omega and divergence on relation semirings in that section.

We first consider trace semirings. By definition, they are complete and satisfy all necessary infinite distributivity laws. Stars can therefore be determined by iteration, omegas cannot.

Sets of traces S over P and A can always be partitioned in its *test part* $S_t = S \cap P$ and its *test-free* or action *part* $S_a = S - P$, i.e., $S = S_t + S_a$. This allows us to calculate S_a^ω separately and then to combine them by Equation (1) to $S^\omega = S_a^\omega + S_a^* S_t \top$.

Since S_a is test-free, every trace $\tau \in S_a \top$ satisfies $|\tau| > 1$. Therefore, by induction, $|\tau| > n$ for all $\tau \in S_a^n \top$ and consequently $S_a^\omega \le \inf(S_a^i \top : i \in \mathbb{N}) = \emptyset$.

As a conclusion, in trace models omega can be explicitly defined by the star. This might be surprising: Omega, which seemingly models infinite iteration, reduces to finite iteration after which a miracle (*anything*) happens. By the results of the previous sections, the argument also applies to language and path semirings. In the case of languages, the argument is known as *Arden's rule* [1]. In particular, the test algebras of language algebras are always $\{\emptyset, \{\varepsilon\}\}$. Therefore $L^\omega = \emptyset$ iff $\varepsilon \notin L$ for every language $L \in 2^{A^*}$.

Theorem 7.1. *Assume an arbitrary element a of $2^{(P,A)^*}$, 2^{A^*} and 2^{P^*}, respectively. Let $a_t = a \cap 1$ denote the test and $a_a = a - a_t$ the action part of a.*

(*a*) *In trace semirings, $a^\omega = (a_a)^* a_t \top$ for any $a \in 2^{(P,A)^*}$.*
(*b*) *In language semirings, $a^\omega = A^*$ if $\varepsilon \in a$ and $\emptyset$ otherwise for any $a \in 2^{A^*}$.*
(*c*) *In path semirings, $a^\omega = a^* a_t \top$ for any $a \in 2^{P^*}$.*

In relation semirings the situation is different: there is no notion of length that would increase through iteration. We will therefore determine omegas in relation semirings relative to a notion of divergence (cf. Section 9).

8 Divergence Semirings

An operation of divergence can be axiomatised algebraically on i-semirings with additional modal operators. The resulting divergence semirings are similar to Goldblatt's *foundational algebras* [6].

An i-semiring S is called *modal* [12] if it can be endowed with a total operation $\langle a \rangle : \mathsf{test}(S) \to \mathsf{test}(S)$, for each $a \in S$, that satisfies the axioms

$$\langle a \rangle p \le q \Leftrightarrow ap \le qa \quad \text{and} \quad \langle ab \rangle p = \langle a \rangle \langle b \rangle p.$$

Intuitively, $\langle a \rangle p$ characterises the set of states with at least one a-successor in p. A *domain* operation $\mathsf{dom} : S \to \mathsf{test}(S)$ is obtained from the diamond operator as $\mathsf{dom}(a) = \langle a \rangle 1$. Alternatively, domain can be axiomatised on i-semirings, even equationally, from which diamonds are defined as $\langle a \rangle p = \mathsf{dom}(ap)$ [3]. The axiomatisation of modal semirings extends to modal Kleene algebras and modal omega algebras without any further modal axioms.

We will use the following properties of diamonds and domain [7]: $\langle p\rangle q = pq$, $\mathsf{dom}(a) = 0 \Leftrightarrow a = 0$, $\mathsf{dom}(\top) = 1$, $\mathsf{dom}(p) = p$. Also, domain is isotone and diamonds are isotone in both arguments.

A modal semiring S is a *divergence semiring* [3] if it has an operation $\nabla : S \to \mathsf{test}(S)$ that satisfies the ∇-unfold and ∇-co-induction axioms

$$\nabla a \leq \langle a\rangle \nabla a \quad \text{and} \quad p \leq \langle a\rangle p \Rightarrow p \leq \nabla a.$$

We call ∇a the *divergence* of a. This axiomatisation can be motivated on trace semirings as follows: The test $p - \langle a\rangle p$ characterises the set of a-maximal elements in p, that is, the set of elements in p from which no further a-action is possible. ∇a therefore has no a-maximal elements by the ∇-unfold axiom and by the ∇-co-induction axiom it is the greatest set with that property. It is easy to see that $\nabla a = 0$ iff a is Noetherian in the usual set-theoretic sense. Divergence therefore comprises the standard notion of program termination. All those states that admit only finite traces are characterised by the complement of ∇a.

The ∇-co-induction axiom is equivalent to $p \leq q + \langle a\rangle p \Rightarrow p \leq \nabla a + \langle a^*\rangle q$, which has the same structure as the omega co-induction axiom. In particular, ∇a is the greatest fixed point of the function $\lambda x.\langle a\rangle x$, which corresponds to a^ω and $\nabla a + \langle a^*\rangle q$ is the greatest fixed point of the function $\lambda x.q + \langle a\rangle x$, which corresponds to $a^\omega + a^*b$. Moreover, the least fixed point of $\lambda x.q + \langle a\rangle x$ is $\langle a^*\rangle q$, which corresponds to a^*b. These fixed points are now defined on test algebras, which are Boolean algebras. Iterative solutions exist again when the test algebra is finite and all diamonds are defined. In general

$$\nabla a \leq \inf(\langle a^i\rangle 1 : i \in \mathbb{N}) = \inf(\mathsf{dom}(a^i) : i \in \mathbb{N}).$$

However, the algebra A_3^2 shows that even finite i-semirings, which always have a complete test algebra, need not be modal semirings (cf. Example 9.2 below).

We will need the properties $\langle a\rangle \nabla a \leq \nabla a, \nabla p = p$ and $\nabla a \leq \mathsf{dom}(a)$ of divergence and isotonicity of ∇ [7].

9 Divergence Across Models

We will now relate omega and divergence in all models presented so far. Concretely, we will investigate the identities $(\nabla a)\top = a^\omega$ and $\nabla a = \mathsf{dom}(a^\omega)$ that arose from our motivating example in Section 3. We will say that omega is *tame* if every a satisfies the first identity; it will be called *benign* if every a satisfies the second one. We will also be interested in the *taming condition* $\mathsf{dom}(a)\top = a\top$. All abstract results of this and the next section has been again automatically verified by Prover9 or Mace4.

First, we consider these properties on relation semirings which we could not treat as special cases of trace semirings in Section 7. It is well known from relation algebra that all relation semirings satisfy the taming condition. We will see in the following section through abstract calculations that omega and divergence are related in relation semirings as expected and, as a special case, $a^\omega = 0$ iff a is Noetherian in relation semirings.

We now revisit the finite i-semirings of Examples 4.1 and 4.2.

Example 9.1. In A_2, $\mathsf{dom}(0) = 0$ and $\mathsf{dom}(1) = 1$. By this, $\nabla 0 = 0$ and $\nabla 1 = 1$.

Example 9.2. In A_3^1 and A_3^3, the test algebra is always $\{0, 1\}$; $\mathsf{dom}(0) = 0$ and $\mathsf{dom}(1) = 1$. Moreover, $\nabla 0 = 0$ and $\nabla 1 = 1$. Setting $\mathsf{dom}(a) = 1 = \nabla a$ turns both into divergence semirings. In contrast, domain cannot be defined on A_3^2.

Consequently, omega is not tame in A_3^2, since $\nabla a \top$ is undefined here, and in A_3^3. However, it is tame in A_3^1 and A_2. In all four finite i-semirings, omega is benign.

Let us now consider trace, language and path semirings. Domain, diamond and divergence can indeed be defined on all these models. On a trace semiring,

$$\mathsf{dom}(S) = \{s : s \in P \text{ and } \exists \tau \in (P, A)^* : s \cdot \tau \in S\}.$$

So, as expected, $\nabla S = \inf(\mathsf{dom}(S^i) : i \in \mathbb{N})$; it characterises all states where infinite paths may start. However, since the omega operator is related to finite behaviour in all these models (cf. Theorem 7.1), the expected relationships to divergence fail.

Lemma 9.3. *The taming condition does not hold on some trace and path semirings. Omega is neither tame nor benign.*

The situation for language semirings, where states are forgotten, is different.

Lemma 9.4.

(*a*) *The taming condition does not hold in some language semirings.*
(*b*) *Omega is tame in all language semirings.*
(*c*) $(\nabla a)\top = a^\omega \not\Rightarrow \mathsf{dom}(a)\top = a\top$ *in some language semirings.*

In the next section we will provide an abstract argument that shows that omega is benign on language semirings (without satisfying the taming condition).

As a conclusion, omega behaves as expected in relation semirings, but not in trace, language and path semirings. This may be surprising: While relations are standard for finite input/output behaviour, traces, languages and paths are standard for infinite behaviour, including reactive and hybrid systems. As we showed before, in these models omega can be expressed by the finite iteration operator and therefore it does not model proper infinite iteration. In contrast to that the divergence operator models infinite behaviour in a natural way.

10 Taming the Omega

Our previous results certainly deserve a model-independent analysis. We henceforth briefly call *omega divergence semirings* a divergence semiring that is also an omega algebra. We will now consider tameness of omega for this class. It is easy to show that the simple identities

$$a\top \leq \mathsf{dom}(a)\top, \quad a^\omega \leq (\nabla a)\top, \quad \mathsf{dom}(a^\omega) \leq \nabla a,$$

hold in all omega divergence semirings [7]. Therefore we only need to consider the relationships between their converses.

Theorem 10.1. *In the class of omega divergence semirings, the following implications hold, but not their converses.*

$$\forall a.\,(\mathsf{dom}(a)\top \leq a\top) \Rightarrow \forall a.\,(\nabla a)\top \leq a^\omega,$$
$$(\nabla a)\top \leq a^\omega \Rightarrow \nabla a \leq \mathsf{dom}(a^\omega).$$

Theorem 10.1 shows that the taming condition implies that omega is tame, which again implies that omega is benign. The fact that omega is benign whenever it satisfies the taming condition has already been proved in [3]. In particular, all relational semirings are tame and benign, since they satisfy the taming condition.

Theorem 10.1 concludes our investigation of divergence and omega. It turns out that these two notions of non-termination are unrelated in general. Properties that seem intuitive for relations can be refuted on three-element or natural infinite models. The taming condition that seems to play a crucial role could only be verified on (finite and infinite) relation semirings.

11 Conclusion

We compared two algebraic notions of non-termination: the omega operator and divergence. It turned out that divergence correctly models infinite behaviour on all models considered, whereas omega shows surprising anomalies. In particular, omega is not benign (whence not tame) on traces and paths, which are among the standard models for systems with infinite behaviour such as reactive and hybrid systems. A particular advantage of our algebraic approach is that this analysis could be carried out in a rather abstract, uniform and simple way.

The main conclusion of this paper, therefore, is that idempotent semirings are a very useful tool for reasoning about termination and infinite behaviour across different models. The notion of divergence is a simple but powerful concept for representing that part of a state space at which infinite behaviour may start. The impact of this concept on the analysis of discrete dynamical systems, in particular by automated reasoning, remains to be explored. The omega operator, however, is appropriate only under some rather strong restrictions which eliminate many models of interest. Our results clarify that omega algebras are generally inappropriate for infinite behaviour: It seems unreasonable to sequentially compose an infinite element a with another element b to ab. Two alternatives to omega algebras allow adding infinite elements: The weak variants of omega algebras introduced by von Wright [16] and elaborated by Möller [11], and in particular the divergence modules introduced in [15], based on work of

Ésik and Kuich [5], in which finite and infinite elements have different sorts and divergence is a mapping from finite to infinite elements. All these variants are developed within first-order equational logic and therefore support the analysis of infinite and terminating behaviours of programs and transition systems by automated deduction [15]. The results of this paper link this abstract analysis with properties of particular models which may arise as part of it..

Acknowledgement. We are grateful to Bernhard Möller for proof-reading.

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Appendices

A A Proof Template for Prover9

```
 op(500, infix, "+").   %addition
 op(490, infix, ";").   %multiplication
 op(480, postfix, "*"). %star
 op(470, postfix, "^"). %omega

 formulas(sos).
% Kleene algebra axioms
     x+y = y+x  &  x+0 = x  &  x+(y+z) = (x+y)+z.
     x;(y;z) = (x;y);z  &  x;1 = x  &  1;x = x.
     0;x = 0  &  x;0 = 0.
     x;(y+z) = x;y+x;z  &  (x+y);z = x;z+y;z.
     x+x = x.
     x <= y <-> x+y = y.
     1+x;x* = x*  &  1+x*;x = x*.
     z+x;y <= y -> x*;z <= y  &  z+y;x <= y -> z;x* <= y.

% Boolean domain axioms (a la Desharnais & Struth)
     a(x);x = 0  &  a(x;y) = a(x;a(a(y)))  &  a(a(x))+a(x) = 1.
     d(x) = a(a(x)).    %domain defined from antidomain

% divergence
     d(x;div(x)) = div(x).
     d(y) <= d(x;d(y))+d(z) -> d(y) <= div(x)+d(x*;z).

% omega axioms
     x;x^ = x^  &  z <= x;z+y -> z <= x^+x*;y.

% additional laws
     T = 1^.
     x <= y -> d(x) <= d(y).
 end_of_list.

 formulas(goals). % for Thm 10.1; to be commented in one by one
   %all x(d(x);T <= x;T) -> all x(div(x);T <= x^).
   %div(x);T <= x^ -> div(x) <= d(x^).
   %all x(d(x);T <= x;T) <- all x(div(x);T = x^).
   %div(x);T <= x^ -> div(x) = d(x^).
 end_of_list.
```

B Proofs

Lemma 6.5. *There is no trace semiring over P and A that is isomorphic to the full relation semiring over a finite set Q with $|Q| > 1$.*

Proof. If there is at least one action in the trace semiring, then the trace semiring is infinite whereas the size of the relation semiring is $2^{|Q|^2}$. Otherwise, all traces will be single states and multiplication will therefore commute on the trace semiring, but not on the relation semiring. Therefore there cannot exist an isomorphism. □

Lemma 9.3. *The taming condition does not hold on some trace and path semirings. Omega is neither tame nor benign.*

Proof. Consider the case of trace semirings. Let $P = \{s\}$ and $A = \{\alpha\}$ and let S be the set consisting of the single trace $s\alpha s$. Then $\mathsf{dom}(S) = \{s\} = \nabla S$ and $\mathsf{dom}(S)\top = \{s\}\top = \nabla(S)\top$ is the set of all non-empty traces over p and α. Moreover, $S\top = \{s.\alpha.\tau : \tau \in (P, A)^*\}$. Finally, Theorem 7.1($a$) implies that $S^\omega = S_a^* S_t \top = \emptyset$ since $S_t = \emptyset$ in the example. This refutes all identities for trace semirings. The argument translates to path semirings by forgetting actions. □

Lemma 9.4.

(a) *The taming condition does not hold in some language semirings.*
(b) *Omega is tame in all language semirings.*
(c) *Tameness does not imply the taming condition in some language semirings.*

Proof. In language semirings the test algebra is $\{\emptyset, \{\varepsilon\}\}$. So $\mathsf{dom}(L) = \{\varepsilon\}$ iff $L \neq 0$ for every $L \in 2^{A^*}$.

(a) Consider the language semiring over the single letter a and the language $L = \{a\}$. Then $\mathsf{dom}(L) = \{\varepsilon\}$ and therefore $\mathsf{dom}(L)\top = \top \neq L\top$, since $\varepsilon \in \top$, but $\varepsilon \notin L\top$.
(b) $\nabla L = \inf(\mathsf{dom}(L^i) : i \in \mathbb{N}\} = \{\varepsilon\}$ iff $L \neq \emptyset$. Therefore $(\nabla L)\top = \top$ iff $L \neq \emptyset$ and $(\nabla L)\top = \emptyset$ iff $L = \emptyset$. It has already been shown in Lemma 7.1(b) that L^ω satisfies the same conditions.
(c) Immediate from (a) and (b). □

Theorem 10.1. *In the class of omega divergence semirings, the following implications hold, but not their converses.*

$$\forall a.\, (\mathsf{dom}(a)\top \leq a\top) \Rightarrow \forall a.\, (\nabla a)\top \leq a^\omega,$$
$$(\nabla a)\top \leq a^\omega \Rightarrow \nabla a \leq \mathsf{dom}(a^\omega).$$

Proof. Both implications can be proved in a few seconds by Prover9 on any personal computer with the input file from Appendix A.

The converse of the first implication fails in the class of language semirings by Lemma 9.4(c).

The converse of the second implication fails in A_3^3 since $\nabla a = 1 = \mathsf{dom}(a) = \mathsf{dom}(a^\omega)$ holds in this model, but $(\nabla a)\top = 1 > a = a^\omega$ by Example 4.2 and 9.2. □