# Cardinality of Relations with Applications

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**Abstract**

Based on Y. Kawahara's characterisation of the cardinality of relations we derive some fundamental properties of cardinalities concerning vectors, points and mapping-related relations. As applications of these results we verify some properties of linear orders and graphs in a calculational manner. These include the cardinalities of rooted trees and some estimates concerning graph parameters. We also calculationally prove the result of D. Kőnig that in bipartite graphs the matching number equals the vertex cover number.

*Keywords:* Relation algebra, cardinality operation, point axiom, decomposition, graph parameter, matching, vertex cover, bipartite relation

## 1. Introduction

Based on pioneering work of mainly G. Boole, A. De Morgan, C.S. Peirce and E. Schröder in the 19th century, the modern axiomatic investigations of the calculus of binary relations started with the seminal paper [28] of A. Tarski on relation algebra in the middle of the 20th century. Since the 1970s this algebraic structure has widely been used by many mathematicians, engineers and computer scientists as a conceptual and methodological base for problem solving in areas like graph theory, theory of orders and lattices, combinatorics, preference and scaling, social choice theory, algorithmics, data bases, and semantics of programming languages. A lot of examples and references to relevant literature can be found e.g., in [2, 8, 10, 11, 25, 27] and the proceedings of the conference series "Relational and Algebraic Methods in Computer Science".

The use of relation algebra brings many advantages: Concerning modelling, it is mainly due to the fact that relations and many objects of discrete mathematics are essentially the same or closely related. For instance, a directed graph is nothing else than a relation on a non-empty and finite set of vertices, and also for other classes of graphs there are simple and elegant ways to model them with relations, as shown in [25, 27], for example. Secondly, the use of relation algebra frequently leads to very precise proofs, where calculational transformations constitute the decisive parts. This has the advantage of clarifying the proof structure frequently, reducing the danger of doing wrong proof steps and to opening the possibility for proof mechanisation, for instance, by automated theorem provers or proof assistants. See e.g., [3, 4, 7, 13, 15] for the latter. Thirdly, the set-theoretic standard model of relation algebra can easily and efficiently be implemented. This supports prototyping and validation tasks in a significant manner, e.g., by the BDD-based special-purpose computer algebra system RelView (see [6]).

Experience has also shown that for advanced applications the "classical" homogeneous relation algebra in the sense of [28] (and further developed in [18, 19, 29], for example) has to be modified. To be able to treat not only relations on one universe but on different sets, in [24] types have been introduced, leading to the notion of a heterogeneous relation algebra. Based on this and following the manner how K.C. Ng and A. Tarski added in [30] the Kleene star as an additional operation for reflexive-transitive closures to homogeneous relation algebra, relational products, sums and embeddings have been axiomatised to deal, for example, with $n$-ary functions, case distinctions and restrictions, respectively. Set-theoretic membership relations and some variants (on function domains) have also been introduced

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in the same way, mainly for the use in relation-algebraic semantics. See [2, 25, 27, 31] for details and references to relevant literature.

In this paper we investigate an extension of heterogeneous relation algebra. We are concerned with a cardinality operation on relations, the axiomatisation of which originates from [16]. In this paper Y. Kawahara acknowledges the considerable influence of [27] to the formal relation-algebraic study of graphs. But he also mentions that "the cardinality of relations is treated rather implicitly or intuitively" [16, Page 251]. Therefore, he develops a cardinality operation on relations and demonstrates by some applications in basic graph theory that its axiomatic specification can be used to reason about cardinalities of relations in a purely calculational and algebraic manner. In [5] the axiomatisation of [16] is applied for the formal assertion-based development and verification of relational approximation algorithms, where cardinalities play an important role when proving the desired approximation bound.

The present paper is a continuation of [16] and [5]. We extend the stock of fundamental properties of cardinalities of relations by several results that concern vectors, points and mapping-related relations. In this regard the point axiom and the decomposition of relations into disjoint unions play an important role. To show the usefulness of the properties, we present some applications. The remainder of the paper is organised as follows: In Section 2 and Section 3 we shortly recall those fundamentals of heterogeneous relation algebra we will need in the following sections; this includes the point axiom and some important consequences. Then, in Section 4, we present Y. Kawahara's axiomatisation of the cardinality operation on relations and some general properties. Specific properties of the cardinality operation with regard to vectors and points and of relations which are related to mappings are presented in Section 5 and Section 6, respectively. Some simple applications that base on these properties are shown in Section 7, e.g., calculational proofs of cardinalities of rooted trees and of some estimates concerning well-known graph parameters. In Section 8 we apply our results to a more complex example. We calculationally prove the theorem of D. Kőnig saying that in bipartite graphs the matching number and the vertex cover number coincide. Section 9 contains some concluding remarks.

## 2. Relation-Algebraic Prerequisites

In this section we recall the fundamentals of relation algebra based on the heterogeneous approach of [24] and further developed especially in [25, 27]. Set-theoretic relations form the standard model of relation algebras. We assume the reader to be familiar with the basic operations on them, viz. $R^{\mathsf{T}}$ (transposition), $\overline{R}$ (complementation), $R \cup S$ (union), $R \cap S$ (intersection), $R;S$ (composition), the predicates $R \subseteq S$ (inclusion) and $R = S$ (equality), and the special relations $\mathsf{O}$ (empty relation), $\mathsf{L}$ (universal relation) and $\mathsf{I}$ (identity relation). Relations of the same type equipped with the Boolean operations, the inclusion and the constants $\mathsf{O}$ and $\mathsf{L}$ form complete Boolean lattices. Some further well-known algebraic properties of relations are $\overline{R^{\mathsf{T}}} = \overline{R}^{\mathsf{T}}$, $(R \cup S)^{\mathsf{T}} = R^{\mathsf{T}} \cup S^{\mathsf{T}}$, $(R \cap S)^{\mathsf{T}} = R^{\mathsf{T}} \cap S^{\mathsf{T}}$, $(R^{\mathsf{T}})^{\mathsf{T}} = R$, $(R;S)^{\mathsf{T}} = S^{\mathsf{T}};R^{\mathsf{T}}$, and the monotonicity of transposition, union, intersection and composition.

The theoretical framework for these laws (and many others) to hold is that of a (heterogeneous) *relation algebra* in the sense of [24, 25, 27], with typed relations as elements. This implies that each relation has a source and a target and we write $R : X \leftrightarrow Y$ to express that $R$ is of type $X \leftrightarrow Y$ with source $X$ and target $Y$. In case of set-theoretic relations $R : X \leftrightarrow Y$ means that $R$ is a subset of the direct product $X \times Y$ and then $X$ and $Y$ are also called *carrier sets*. As constants and operations of a relation algebra we have those of set-theoretic relations, where we frequently overload the symbols $\mathsf{O}$, $\mathsf{L}$ and $\mathsf{I}$, i.e., avoid the binding of types to them. Only when helpful or necessary we use indices to annotate types such as $\mathsf{L}_{XY}$ for the universal relation of type $X \leftrightarrow Y$ and $\mathsf{I}_X$ for the identity relation of type $X \leftrightarrow X$. The axiomatisation of relation algebra we will present now follows [25, 27].

**Axioms 2.1** (Relation Algebra)**.** *The following hold:*

*(R1) For all types $X \leftrightarrow Y$ the relations of type $X \leftrightarrow Y$ constitute a complete Boolean lattice under the Boolean operations, the inclusion, the empty relation and the universal relation.*

*(R2) Composition of relations is associative and the identity relations are neutral elements with respect to composition.*

*(R3) For all relations $Q$, $R$ and $S$ (with appropriate types) the three inclusions $Q;R \subseteq S$, $Q^{\mathsf{T}};\overline{S} \subseteq \overline{R}$ and $\overline{S};R^{\mathsf{T}} \subseteq \overline{Q}$ are equivalent.*

*(R4) For all relations R and all universal relations (with appropriate types) from* $R \neq \mathsf{O}$ *it follows* $\mathsf{L};R;\mathsf{L} = \mathsf{L}$.

In [27] the equivalences of (R3) are called the *Schröder rules* and the implication of (R4) is called the *Tarski rule*. In the relation-algebraic proofs of this paper we will mention only applications of (R3), (R4) and "non-obvious" consequences of the axioms, like the inclusion

$$Q;R \cap S \subseteq (Q \cap S;R^{\mathsf{T}});(R \cap Q^{\mathsf{T}};S), \tag{1}$$

for all relations $Q : X \leftrightarrow Y$, $R : Y \leftrightarrow Z$ and $S : X \leftrightarrow Z$, in [27] called the *Dedekind rule*. Additionally, we use the following rule for all relations $R$

$$R^{\mathsf{T}};\overline{\overline{R};\mathsf{L}} = \mathsf{O}\ , \tag{2}$$

which is a consequence of the Schröder equivalences, because $R;\mathsf{L} \subseteq R;\mathsf{L}$ and $\overline{\mathsf{L}} = \mathsf{O}$ hold. Furthermore, we will assume that complementation and transposition bind stronger than composition, composition binds stronger than union and intersection, and that all expressions and formulae are well-typed. The latter assumption allows to suppress many type annotations, since types of relations can be derived from others with given types using the typing rules of the relational operations.

In the following we recapitulate some well-known classes of relations used in the remainder of this paper and specify them in an algebraic way. For more details on such an approach (including the relation-algebraic specification of further important classes) see again [25, 27].

A relation $R$ is *univalent* iff $R^{\mathsf{T}};R \subseteq \mathsf{I}$, and *total* iff $R;\mathsf{L} = \mathsf{L}$ (for all universal relations with appropriate types), where totality of $R$ is equivalent to $\mathsf{I} \subseteq R;R^{\mathsf{T}}$. A *mapping* is a univalent and total relation. For a univalent $R$ we have $R;\overline{S} \subseteq \overline{R;S}$ and for a total $R$ we have $R;\overline{S} \supseteq \overline{R;S}$, for all $S$. The relation $R$ is *injective* iff $R^{\mathsf{T}}$ is univalent and *surjective* iff $R^{\mathsf{T}}$ is total. Hence, if $S$ is injective, then $\overline{R};S \subseteq \overline{R;S}$, and if $S$ is surjective, then $\overline{R};S \supseteq \overline{R;S}$, for all $R$. Additionally, given a univalent relation $R : X \leftrightarrow Y$ and $S, T : Y \leftrightarrow Z$, we have

$$R;(S \cap T) = R;S \cap R;T\ . \tag{3}$$

Note that in general only $R;(S \cap T) \subseteq R;S \cap R;T$ holds. However, composition distributes over (arbitrary) unions.

Relations of type $X \leftrightarrow X$ are *homogeneous*. Let $R$ be homogeneous. Then $R$ is *reflexive* iff $\mathsf{I} \subseteq R$, *irreflexive* iff $R \subseteq \overline{\mathsf{I}}$, *symmetric* iff $R = R^{\mathsf{T}}$, *antisymmetric* iff $R \cap R^{\mathsf{T}} \subseteq \mathsf{I}$, and *transitive* iff $R;R \subseteq R$. A reflexive, antisymmetric and transitive relation $R$ is a *partial order relation* and if additionally $R \cup R^{\mathsf{T}} = \mathsf{L}$ holds then $R$ is a *linear order relation*. The least transitive relation containing $R$ is its *transitive closure* $R^+ = \bigcup_{k \geq 1} R^k$ and the least reflexive and transitive relation containing $R$ is its *reflexive-transitive closure* $R^* = \bigcup_{k \geq 0} R^k$, where $R^0 := \mathsf{I}$ and $R^k := R;R^{k-1}$, for all $k > 0$. For these constructions we have $R^* = R^+ \cup \mathsf{I}$ and $R^+ = R;R^* = R^*;R$. If $R^+$ is irreflexive, then $R$ is *cycle-free*.

A (relational) *vector* is a relation $v$ with $v = v;\mathsf{L}$. Usually vectors are denoted by lower-case letters. For a set-theoretic vector $v : X \leftrightarrow Y$ the condition $v = v;\mathsf{L}$ means that $v$ can be written in the form $v = Z \times Y$ with a subset $Z$ of $X$. Then we say that *$v$ models the subset $Z$ of $X$*. For this purpose the target of a vector is irrelevant. Therefore, we often use the specific singleton set $\mathbb{1}$ as target. For algebraically dealing with $\mathbb{1}$, e.g., if automated theorem provers or proof assistants are used, the following axiom proved to be sufficient.

**Axiom 2.2** (Singleton Set Axioms)**.** *It holds* $\mathsf{I}_{\mathbb{1}} \neq \mathsf{O}_{\mathbb{1}\mathbb{1}}$ *and also* $\mathsf{I}_{\mathbb{1}} = \mathsf{L}_{\mathbb{1}\mathbb{1}}$.

In the Boolean matrix model of set-theoretic relations vectors correspond to row-constant Boolean matrices, i.e., matrices with only 1-entries or only 0-entries in each row. Thus, a vector of type $X \leftrightarrow \mathbb{1}$ corresponds to a Boolean column vector. Because of Axiom 2.2 vectors of the latter type are univalent. A further consequence of Axiom 2.2 is that, together with the Tarski rule, we get for all sets $X$ and $Y$ the equation

$$\mathsf{L}_{X\mathbb{1}};\mathsf{L}_{Y\mathbb{1}}{}^{\mathsf{T}} = \mathsf{L}_{X\mathbb{1}};\mathsf{I}_{\mathbb{1}};\mathsf{L}_{Y\mathbb{1}}{}^{\mathsf{T}} = \mathsf{L}_{X\mathbb{1}};\mathsf{I}_{\mathbb{1}};\mathsf{L}_{\mathbb{1}Y} = \mathsf{L}_{XY}\ . \tag{4}$$

A (relational) *point* is a vector $p$ such that $p \neq \mathsf{O}$ and $p;p^{\mathsf{T}} \subseteq \mathsf{I}$. It is easy to see that a set-theoretic point $p : X \leftrightarrow Y$ models a singleton subset of the set $X$ and it corresponds in the case $Y = \mathbb{1}$ to a Boolean column vector with a single 1-entry. If $\{x\}$ is modelled by $p$, then we say that *$p$ models the element $x \in X$*. For points $p$ and $q$ we have $p;q^{\mathsf{T}} \neq \mathsf{O}$, and that $p \subseteq q$, $p = q$, $p;q^{\mathsf{T}} \subseteq \mathsf{I}$ and $p^{\mathsf{T}};q = \mathsf{L}$ are equivalent. Moreover, $p \neq q$, $p \cap q = \mathsf{O}$, $p;q^{\mathsf{T}} \subseteq \overline{\mathsf{I}}$ and $p^{\mathsf{T}};q = \mathsf{O}$ are equivalent as well. Furthermore, we have that $p;q^{\mathsf{T}} \subseteq R$ if and only if $p \subseteq R;q$, for all $R$. Points are surjective and, as a consequence, the transpose of a point $p$ is a mapping. Finally, points constitute atoms (upper neighbors of $\mathsf{O}$) in the set of all vectors. In particular, this yields that the intersection of two different points is $\mathsf{O}$.

## 3. The Point Axiom and Some Consequences

In this paper we only consider set-theoretic relations. However, we do not use point-wise arguments, that is, the notation $(x, y) \in R$. Instead, we treat set-theoretic relations in a calculational and purely algebraic manner only. This approach is based on some fundamental properties of their operations $\cup, \cap, ;, \overline{\phantom{x}}$ and $^{\mathsf{T}}$, the predicates $=$ and $\subseteq$ and the constant relations $\mathsf{O}, \mathsf{L}$ and $\mathsf{I}$, which are taken as axioms. So far, we have introduced the axioms (R1) to (R4) of a relation algebra and the singleton set axiom for specifying the meaning of $\mathbb{1}$. Now we introduce a further axiom, where, for a given vector $v$, we denote by $\mathcal{P}(v)$ the set of all points $p$ such that $p \subseteq v$.

**Axiom 3.1** (Point Axiom)**.** *For all sets X it holds* $\mathsf{L}_{X\mathbb{1}} = \bigcup_{p\in\mathcal{P}(\mathsf{L}_{X\mathbb{1}})} p$.

In the literature different versions of the point axiom can be found. Our version stems from [14]. In words Axiom 3.1 states that each universal vector with target $\mathbb{1}$ is the (disjoint) union of the points it contains, i.e., can be decomposed into these points. It holds for set-theoretic relations. In the remainder of the paper we assume Axiom 3.1 as additional axiom. Besides relation-algebraic properties implied by the axioms (R1) to (R4) and the singleton set axiom, therefore, we also are allowed to use some that assume Axiom 3.1 to be satisfied. Lemma 3.1 presents two such properties. The first one generalises Axiom 3.1 to all vectors with target $\mathbb{1}$ and the second one says that each identity relation can be represented as the union of compositions of points with their transposes. These compositions are atoms in the lattice-theoretical sense, as shown in [27], and thus all elements of the union of Lemma 3.1.(ii) are again pairwise disjoint.

**Lemma 3.1.** *(i) For all vectors* $v : X \leftrightarrow \mathbb{1}$ *we have* $v = \bigcup_{p\in\mathcal{P}(v)} p$.

*(ii) For all identity relations* $\mathsf{I} : X \leftrightarrow X$ *we have* $\mathsf{I} = \bigcup_{p\in\mathcal{P}(\mathsf{L}_{X\mathbb{1}})} p;p^{\mathsf{T}}$.

The lemma is shown in [14] as part of Proposition 3.3.4. An immediate consequence of the first statement of Lemma 3.1 is that $v \neq \mathsf{O}$ implies $\mathcal{P}(v) \neq \emptyset$ and, hence, we have the following fact.

**Lemma 3.2.** *Each non-empty vector contains a point.*

This lemma allows to prove that for each non-empty relation $R$ there exist points $p$ and $q$ such that $p;q^{\mathsf{T}} \subseteq R$, i.e., that the point axiom introduced in [26] and used in [25, 27] is satisfied. From that point axiom the *intermediate point theorem* of [27] follows, saying that for all relations $R$ and $S$ and points $p$ and $q$, if $p \subseteq R;S;q$ then there exists a point $r$ such that $p \subseteq R;r$ and $r \subseteq S;q$. Later, we will use the following specific version.

**Theorem 3.1.** *For all relations* $R : X \leftrightarrow Y$*, vectors* $v : Y \leftrightarrow \mathbb{1}$ *and points* $p : X \leftrightarrow \mathbb{1}$*, if* $p \subseteq R;v$*, then there exists a point* $r : Y \leftrightarrow \mathbb{1}$ *with* $p;r^{\mathsf{T}} \subseteq R$ *and* $r \subseteq v$.

*Proof.* Because of the target $\mathbb{1}$ of the vector $v$, the assumption $p \subseteq R;v$ is equivalent to $p \subseteq R;v;\mathsf{I}_{\mathbb{1}}$. Axiom 2.2 imply that $\mathsf{I}_{\mathbb{1}}$ is a point. Hence, from the intermediate point theorem it follows the existence of a point $r : Y \leftrightarrow \mathbb{1}$ with $p \subseteq R;r$ and $r \subseteq v;\mathsf{I}_{\mathbb{1}}$, that is, with $p;r^{\mathsf{T}} \subseteq R$ and $r \subseteq v$. □

Furthermore, Lemma 3.2 yields for all vectors $v$ and points $p$ that either $p \subseteq v$ or $p \subseteq \overline{v}$ holds. We use this property to prove the following result.

**Lemma 3.3.** *For all relations R and points p we have* $p \subseteq R;\mathsf{L}$ *if and only if* $R^{\mathsf{T}};p \neq \mathsf{O}$.

*Proof.* Using that $R;\mathsf{L}$ is a vector and $p$ is a point in the first and one of the Schröder rules in the last step we have the following equivalences:

$$p \not\subseteq R;\mathsf{L} \iff p \subseteq \overline{R;\mathsf{L}} \iff R;\mathsf{L} \subseteq \overline{p} \iff R^{\mathsf{T}};p \subseteq \mathsf{O}\,. \qquad \square$$

From the second statement of Lemma 3.1 we obtain the following decomposition result that will play an important role in the remainder of the paper.

**Theorem 3.2.** *For all relations* $R : X \leftrightarrow Y$ *we have* $R = \bigcup_{p\in\mathcal{P}(\mathsf{L}_{Y\mathbb{1}})} R;p;p^{\mathsf{T}}$*, where all relations of the union are pairwise disjoint and univalent.*

*Proof.* With the help of Lemma 3.1.(ii) we get

$$R = R;\mathsf{I} = R;\bigcup_{p\in\mathcal{P}(\mathsf{L}_{Y\mathbb{1}})} p;p^{\mathsf{T}} = \bigcup_{p\in\mathcal{P}(\mathsf{L}_{Y\mathbb{1}})} R;p;p^{\mathsf{T}} .$$

To show that the relations of the union are pairwise disjoint, let $p, q : Y \leftrightarrow \mathbb{1}$ be points such that $p \neq q$ holds. Using the Dedekind rule (1) and $q^{\mathsf{T}};p = \mathsf{O}$ we get

$$R;p;p^{\mathsf{T}} \cap R;q;q^{\mathsf{T}} \subseteq (R \cap R;q;q^{\mathsf{T}};p;p^{\mathsf{T}});(p;p^{\mathsf{T}} \cap R^{\mathsf{T}};R;q;q^{\mathsf{T}}) = \mathsf{O} .$$

Finally, to show that all relations of the union are univalent, we calculate for an arbitrary point $p : Y \leftrightarrow \mathbb{1}$ that

$$(R;p;p^{\mathsf{T}})^{\mathsf{T}};R;p;p^{\mathsf{T}} = p;p^{\mathsf{T}};R^{\mathsf{T}};R;p;p^{\mathsf{T}} \subseteq p;\mathsf{L};p^{\mathsf{T}} = p;p^{\mathsf{T}} \subseteq \mathsf{I} ,$$

thereby using the vector property and the injectivity of points. □

If the point $p : Y \leftrightarrow \mathbb{1}$ models the element $y \in Y$, then in the Boolean matrix model of relations the $y$-column of $R;p;p^{\mathsf{T}} : X \leftrightarrow Y$ coincides with the $y$-column of $R : X \leftrightarrow Y$ and all other columns of $R;p;p^{\mathsf{T}}$ are empty. In the same way the "row-oriented" version $R = \bigcup_{p\in\mathcal{P}(\mathsf{L}_{X\mathbb{1}})} p;p^{\mathsf{T}};R$ of Theorem 3.2 can be shown.

## 4. Kawahara's Cardinality of Relations

In [16] Y. Kawahara discusses the cardinality of set-theoretic relations. The main results are a formula, called Dedekind inequality, that allows a calculational treatment of cardinalities of compositions of relations and, based on it, a characterisation of the usual set-theoretic cardinality of finite relations (Theorem 2 of [16]). If the properties of this characterisation are considered as axiomatic specification of a cardinality operation that assigns to all finite relations $R$ a natural number $|R|$ as the cardinality of $R$, then this leads to the following axioms.

**Axioms 4.1** (Cardinality Axioms)**.** *For all finite relations $Q, R$ and $S$ with appropriate types it holds:*

*(C1)* $|R| = 0$ *if and only if* $R = \mathsf{O}$.

*(C2)* $|R| = |R^{\mathsf{T}}|$.

*(C3)* $|R \cup S| = |R| + |S| - |R \cap S|$.

*(C4) If $Q$ is univalent, then* $\max\left\{ \left|R \cap Q^{\mathsf{T}};S\right|, \left|Q \cap S;R^{\mathsf{T}}\right| \right\} \leq |Q;R \cap S|$.

*(C5)* $|\mathsf{I}_{\mathbb{1}}| = 1$.

Axiom (C5) says that the identity relation on the singleton set $\mathbb{1}$ consists of precisely one pair. The original version of Axiom (C4) is formulated as two separate estimates, because both can be considered in the context of infinite relations as well. In this context, $|R| \leq |S|$ means that there is an injective function (in the classical sense) from $R$ to $S$, where $R$ and $S$ are viewed as sets. In fact, Kawahara proves that both estimates are true for arbitrary concrete relations; see [16, Theorem 1]. However, since we deal with finite relations only, we combine the two estimates into a single one for brevity. To simplify the presentation and to avoid additional pre-conditions in lemmas and theorems, we assume the following convention that suffices for our applications for the remainder of the paper.

**Convention 4.1.** *In case of an expression $|R|$ the sets of the type of the relation $R$ are assumed to be finite such that $|R|$ is defined.*

From the two cardinality axioms (C1) and (C3) we get that $R \subseteq S$ implies $|R| \leq |S|$, for all $R$ and $S$, i.e., that the cardinality operation is *monotonic*. Even *strict monotonicity* holds, that is, $R \subset S$ implies $|R| < |S|$, for all $R$ and $S$. A further consequence of (C1) and (C3) is that for all $R$ and $S$ it holds $|S \cap \overline{R}| = |S| - |R|$ if $R \subseteq S$. Finally, these axioms imply $|\bigcup_{R\in\mathcal{R}} R| = \sum_{R\in\mathcal{R}} |R|$, for all finite sets $\mathcal{R}$ of pairwise disjoint relations.

Based on the cardinality axioms (C1) to (C5), in [16] many algebraic laws for the cardinality operation are derived in a purely calculational manner. In the remainder of this paper we only need the following ones (for a proof, see [16], Corollary 1(a) and (c)).

**Lemma 4.1.** *For all relations $Q$, $R$ and $S$ we have:*

*(i)* *If $R$ and $S$ are univalent, then* $|R;S \cap Q| = \left|R \cap Q;S^{\mathsf{T}}\right|$.

*(ii)* *If $R$ is univalent and $S$ is a mapping, then* $|R;S| = |R|$.

A consequence of Lemma 4.1.(ii) is the following refinement of the Decomposition Theorem 3.2, that takes the cardinality of $R : X \leftrightarrow Y$ and of the relations $R;p;p^{\mathsf{T}}$ for all points $p : Y \leftrightarrow \mathbb{1}$ into account.

**Theorem 4.1.** *For all relations $R : X \leftrightarrow Y$ we have* $|R| = \sum_{p \in \mathcal{P}(\mathsf{L}_{Y\mathbb{1}})} |R;p|$.

*Proof.* We calculate

$$|R| = \left|\bigcup_{p \in \mathcal{P}(\mathsf{L}_{Y\mathbb{1}})} R;p;p^{\mathsf{T}}\right| = \sum_{p \in \mathcal{P}(\mathsf{L}_{Y\mathbb{1}})} \left|R;p;p^{\mathsf{T}}\right| = \sum_{p \in \mathcal{P}(\mathsf{L}_{Y\mathbb{1}})} |R;p|,$$

using Theorem 3.2, the axioms (C1) and (C3) (the relations of the union are pairwise disjoint) and that for all points $p : Y \leftrightarrow \mathbb{1}$ the vector $R;p : X \leftrightarrow \mathbb{1}$ is univalent and $p^{\mathsf{T}} : \mathbb{1} \leftrightarrow Y$ is a mapping, since this implies $|R;p;p^{\mathsf{T}}| = |R;p|$ because of Lemma 4.1.(ii). □

Assume again that the point $p : Y \leftrightarrow \mathbb{1}$ models the element $y \in Y$. Then in the Boolean matrix model the $y$-column of $R$ and of $R;p;p^{\mathsf{T}}$ coincide and are equal to the Boolean vector model of $R;p : X \leftrightarrow \mathbb{1}$. So, in Boolean matrix terminology the theorem says that the cardinality of $R$ is the sum of the cardinalities of its columns. Transposing the relation $R$ and using axiom (C2) we get from Theorem 4.1 that $|R| = |R^{\mathsf{T}}| = \sum_{p \in \mathcal{P}(\mathsf{L}_{X\mathbb{1}})} |R^{\mathsf{T}};p|$, i.e., that the cardinality of $R$ is also the sum of the cardinalities of its rows.

Finally, we need one particular property of the relational cardinality. The intuition behind this property is the following one: for every concrete function $f : X \to Y$ and every $A \subseteq X$ one has $|f(A)| \leq |A|$ and if $f$ is injective, then also $|f(A)| = |A|$ holds. When we translate this into relational terms, then $f(A)$ becomes $f^{\mathsf{T}};A$, where now $A$ is a relation. Additionally, one can omit the totality of $f$ and require that $A$ is contained in the domain of $f$ instead. This yields the following result, which constitutes a stronger version of Y. Kawahara's [16] Corollary 1.(g).

**Theorem 4.2.** *For all relations $F : X \leftrightarrow Y$ and $A : X \leftrightarrow Z$ the following holds:*

*(i)* *If $F$ is univalent, then* $\left|F^{\mathsf{T}};A\right| \leq |A|$.

*(ii)* *If $A \subseteq F;\mathsf{L}$ holds and $F$ is injective, then we have* $|A| \leq \left|F^{\mathsf{T}};A\right|$.

*(iii)* *If $A \subseteq F;\mathsf{L}$ holds and $F$ is univalent and injective, then* $\left|F^{\mathsf{T}};A\right| = |A|$.

*Proof.* (i) Suppose that $F$ is univalent. Then we get:

$$\begin{aligned} \left|F^{\mathsf{T}};A\right| &= \left|F^{\mathsf{T}};A \cap F^{\mathsf{T}};A\right| & \\ &\leq \left|F;F^{\mathsf{T}};A \cap A\right| & \text{by axiom (C4)} \\ &\leq |A| & \text{cardinality is monotonic .} \end{aligned}$$

(ii) Now suppose that $A \subseteq F;\mathsf{L}$ holds and that $F$ is injective. Then we have:

$$\begin{aligned} |A| &= |A \cap F;\mathsf{L}| & \text{because of } A \subseteq F;\mathsf{L} \\ &= \left|A \cap \left(F^{\mathsf{T}}\right)^{\mathsf{T}};\mathsf{L}\right| & \\ &\leq \left|F^{\mathsf{T}};A \cap \mathsf{L}\right| & \text{by axiom (C4)} \\ &= \left|F^{\mathsf{T}};A\right| \ . & \end{aligned}$$

(iii) This property is obvious from Theorem 4.2.(ii) and Theorem 4.2.(i). □

## 5. Cardinality Properties of Vectors and Points

In this section we investigate cardinality properties with respect to vectors and points. First, we consider the cardinality of points with the singleton set $\mathbb{1}$ as target. Here we get the following result that is the base for many further facts.

**Lemma 5.1.** *For all points $p : X \leftrightarrow \mathbb{1}$ we have $|p| = 1$.*

*Proof.* The statement follows from

$$|p| = \left|p^{\mathsf{T}}\right| = \left|\mathsf{I}_{\mathbb{1}};p^{\mathsf{T}}\right| = |\mathsf{I}_{\mathbb{1}}| = 1 \ ,$$

using axiom (C2), Lemma 4.1.(ii) (the identity relation $\mathsf{I}_{\mathbb{1}}$ is univalent and the transposed point $p^{\mathsf{T}} : \mathbb{1} \leftrightarrow X$ is a mapping) and axiom (C5). □

Lemma 5.1 allows to prove that the cardinality of a vector with target $\mathbb{1}$ equals the cardinality of the set of all points it contains. In the corresponding Lemma 5.2 the axiomatic relational cardinality and the usual set-theoretic cardinality (also denoted by the vertical bars) are connected.

**Lemma 5.2.** *For all vectors $v : X \leftrightarrow \mathbb{1}$ we have $|v| = |\mathcal{P}(v)|$.*

*Proof.* Because of Lemma 3.1.(i), axiom (C1) and axiom (C3) (the points of the set $\mathcal{P}(v)$ are pairwise disjoint) and Lemma 5.1 we obtain the claim by

$$|v| = |\textstyle\bigcup_{p\in\mathcal{P}(v)} p| = \sum_{p\in\mathcal{P}(v)} |p| = \sum_{p\in\mathcal{P}(v)} 1 = |\mathcal{P}(v)| \ . \qquad \square$$

The composition $v;w^{\mathsf{T}}$ of two vectors $v$ and $w$ forms a rectangle in the relational sense of [25]. As an application of Lemma 5.2 we now prove how to compute the cardinality of this rectangle if $\mathbb{1}$ is the vector's target.

**Theorem 5.1.** *For all vectors $v : X \leftrightarrow \mathbb{1}$ and $w : Y \leftrightarrow \mathbb{1}$ we have $|v;w^{\mathsf{T}}| = |v|\cdot|w|$.*

*Proof.* We calculate as follows:

$$\begin{aligned}
\left|v;w^{\mathsf{T}}\right| &= \left|v;\left(\textstyle\bigcup_{p\in\mathcal{P}(w)} p\right)^{\mathsf{T}}\right| && \text{by Lemma 3.1.(i)}\\
&= \left|\textstyle\bigcup_{p\in\mathcal{P}(w)} v;p^{\mathsf{T}}\right| \\
&= \textstyle\sum_{p\in\mathcal{P}(w)} \left|v;p^{\mathsf{T}}\right| && \text{by axioms } (C1), (C3), \text{see below}\\
&= \textstyle\sum_{p\in\mathcal{P}(w)} |v| && \text{by Lemma 4.1.(ii), see below}\\
&= |v|\cdot|\mathcal{P}(w)| \\
&= |v|\cdot|w| && \text{by Lemma 5.2}\ .
\end{aligned}$$

To show that the union $\bigcup_{p\in\mathcal{P}(w)} v;p^{\mathsf{T}}$ of the second line of the calculation is disjoint, assume $p, q \in \mathcal{P}(w)$ such that $p \neq q$. Since $v^{\mathsf{T}};v : \mathbb{1} \leftrightarrow \mathbb{1}$, we have $v^{\mathsf{T}};v \subseteq \mathsf{L}_{\mathbb{1}\mathbb{1}} = \mathsf{I}_{\mathbb{1}}$ by Axiom 2.2 and thus $v$ is univalent. Then we get

$$v;p^{\mathsf{T}} \cap v;q^{\mathsf{T}} = v;(p^{\mathsf{T}} \cap q^{\mathsf{T}}) = v;(p \cap q)^{\mathsf{T}} = v;\mathsf{O} = \mathsf{O} \ ,$$

using the rule (3) and the fact that $p \cap q = \mathsf{O}$. Finally, Lemma 4.1.(ii) is applicable since the vector $v : X \leftrightarrow \mathbb{1}$ is univalent and each transposed point $p^{\mathsf{T}} : \mathbb{1} \leftrightarrow Y$ is a mapping. □

For each set $X$ the two functions $x \mapsto \{x\} \times \mathbb{1}$ and $p \mapsto \{x \mid p = \{x\} \times \mathbb{1}\}$ constitute a one-to-one correspondence between the set $X$ and the set $\mathcal{P}(\mathsf{L}_{X\mathbb{1}})$ of points in the usual set-theoretic sense, such that we have $|X| = |\mathcal{P}(\mathsf{L}_{X\mathbb{1}})|$. As a consequence, Lemma 5.2 yields $|X| = |\mathcal{P}(\mathsf{L}_{X\mathbb{1}})| = |\mathsf{L}_{X\mathbb{1}}|$. In combination with Theorem 5.1 and Lemma 4.1.(ii), respectively, this allows to describe the cardinalities of universal relations and identity relations in terms of the cardinalities of their carrier sets, as we will show next.

**Lemma 5.3.** *(i) For all sets $X$ and $Y$ we have $|\mathsf{L}_{XY}| = |X|\cdot|Y|$.*

*(ii) For all sets $X$ we have $|\mathsf{I}_X| = |X|$.*

*Proof.* (i) Using property (4) in the first and Theorem 5.1 in the second step we get

$$|\mathsf{L}_{XY}| = \left|\mathsf{L}_{X\mathbb{1}};\mathsf{L}_{Y\mathbb{1}}^{\mathsf{T}}\right| = |\mathsf{L}_{X\mathbb{1}}|\cdot|\mathsf{L}_{Y\mathbb{1}}| = |X|\cdot|Y| \,.$$

(ii) Lemma 4.1.(ii) (the relation $\mathsf{I}_X$ is univalent and $\mathsf{L}_{X\mathbb{1}}$ is a mapping) yields

$$|\mathsf{I}_X| = |\mathsf{I}_X;\mathsf{L}_{X\mathbb{1}}| = |\mathsf{L}_{X\mathbb{1}}| = |X| \,.$$ □

If we combine axiom (C5) with Lemma 5.3.(ii), then we get that $\mathbb{1}$ is in fact a singleton set, since $1 = |\mathsf{I}_{\mathbb{1}}| = |\mathbb{1}|$. At this place it should also be noted that in case of automated theorem provers or proof assistants the equation

$$|\mathsf{L}_{X\mathbb{1}}| = |X| \tag{5}$$

can be used as an additional cardinality axiom to connect the axiomatic relational cardinality with the usual set-theoretic cardinality. But the introduction of such an axiom (5) within a tool requires that sources and targets of relations can no longer be taken as identifiers without any further meaning, but have to be from a type for sets with an own cardinality operation. The relation-algebra library for the proof assistant Coq, presented in [22] and available via the web [21], provides both possibilities for typing relations and thus allows mechanised proofs of results like Lemma 5.2 and Lemma 5.3.

For all relations $R : X \leftrightarrow Y$ we have $R \cup \overline{R} = \mathsf{L}$ and $R \cap \overline{R} = \mathsf{O}$. From these properties, the axioms (C1), (C3) and the first statement of Lemma 5.3 we get for the complement that $|\overline{R}| = |X|\cdot|Y| - |R|$. In the specific case of a point $p : X \leftrightarrow \mathbb{1}$ this, $|\mathbb{1}| = 1$ and Lemma 5.1 yield $|\overline{p}| = |X|\cdot|\mathbb{1}| - |p| = |X| - 1$.

Given a relation $R : X \leftrightarrow Y$, the vector $dom(R) := R;\mathsf{L}_{Y\mathbb{1}} : X \leftrightarrow \mathbb{1}$ denotes the *domain* of $R$ and the vector $ran(R) := R^{\mathsf{T}};\mathsf{L}_{X\mathbb{1}} : Y \leftrightarrow \mathbb{1}$ denotes the *range* of $R$. Since the vector $\mathsf{L}_{X\mathbb{1}}$ is univalent, the axiom (C4), property (4) and the monotonicity of the cardinality operation yield the estimate

$$|dom(R)| = \left|\mathsf{L}_{X\mathbb{1}} \cap R;\left(\mathsf{L}_{Y\mathbb{1}}^{\mathsf{T}}\right)^{\mathsf{T}}\right| \le \left|\mathsf{L}_{X\mathbb{1}};\mathsf{L}_{Y\mathbb{1}}^{\mathsf{T}} \cap R\right| = |\mathsf{L} \cap R| = |R| \,. \tag{6}$$

Similar to (6) the estimate $|ran(R)| \le |R|$ can be shown. The following result presents an estimate of the cardinality of a composition of relations by the cardinalities of domain and range, respectively.

**Theorem 5.2.** *For all relations $R$ and $S$ we have $|R;S| \le |dom(R)|\cdot|ran(S)|$.*

*Proof.* Assuming $Y$ as target of $R$, the claim follows from

$$|R;S| \le |R;\mathsf{L};S| = \left|R;\mathsf{L}_{Y\mathbb{1}};(S^{\mathsf{T}};\mathsf{L}_{Y\mathbb{1}})^{\mathsf{T}}\right| = |R;\mathsf{L}_{Y\mathbb{1}}|\cdot\left|S^{\mathsf{T}};\mathsf{L}_{Y\mathbb{1}}\right| = |dom(R)|\cdot|ran(S)|$$

by means of the monotonicity of the cardinality operation, property (4), Theorem 5.1 and the definition of $dom(R)$ and $ran(S)$. □

Taking both $R$ and $S$ as the same homogeneous universal relation implies that the estimate of Theorem 5.2 is sharp.

When sets are modelled by subrelations of identity relations instead of vectors, then for a relation $R : X \leftrightarrow Y$ the relations $\mathsf{I} \cap R;R^{\mathsf{T}} : X \leftrightarrow X$ and $\mathsf{I} \cap R^{\mathsf{T}};R : Y \leftrightarrow Y$ take over the role of $dom(R)$ and $ran(R)$, respectively. In view of cardinalities there is no difference between these specifications as we show in the following.

**Theorem 5.3.** *For all relations $R : X \leftrightarrow Y$ we have $|dom(R)| = |\mathsf{I} \cap R;R^{\mathsf{T}}|$ and $|ran(R)| = |\mathsf{I} \cap R^{\mathsf{T}};R|$.*

*Proof.* For a proof of the first equation we start with the auxiliary calculation that uses the Dedekind rule (1) in the second step

$$\mathsf{I} \cap R;\mathsf{L} = R;\mathsf{L} \cap \mathsf{I} \cap \mathsf{I} \subseteq (R \cap \mathsf{I};\mathsf{L}^{\mathsf{T}});(\mathsf{L} \cap R^{\mathsf{T}};\mathsf{I}) \cap \mathsf{I} \subseteq R;R^{\mathsf{T}} \cap \mathsf{I} \subseteq \mathsf{I} \cap R;\mathsf{L} \,.$$

This shows $\mathsf{I} \cap R;\mathsf{L} = \mathsf{I} \cap R;R^{\mathsf{T}}$. Now the desired equation is proved by the calculation

$$\begin{aligned}
|dom(R)| &= |R;\mathsf{L}_{Y\mathbb{1}}| \\
&= |\mathsf{I};\mathsf{L}_{X\mathbb{1}} \cap R;\mathsf{L}_{Y\mathbb{1}}| && \text{identity relations are total} \\
&= \left|\mathsf{I} \cap R;\mathsf{L}_{Y\mathbb{1}};\mathsf{L}_{X\mathbb{1}}^{\mathsf{T}}\right| && \text{by Lemma 4.1.(i) (I and } \mathsf{L}_{X\mathbb{1}} \text{ univalent)} \\
&= |\mathsf{I} \cap R;\mathsf{L}| \\
&= \left|\mathsf{I} \cap R;R^{\mathsf{T}}\right| && \text{see above .}
\end{aligned}$$

If we replace in this equation $R$ with $R^{\mathsf{T}}$, then $|ran(R)| = |dom(R^{\mathsf{T}})| = \left|\mathsf{I} \cap R^{\mathsf{T}};R\right|$ follows. □

## 6. Cardinality Properties of Mapping-Related Relations

Theorem 3.2 shows that each relation $R : X \leftrightarrow Y$ can be decomposed into the disjoint union of $|Y|$ univalent subrelations. In this section we first prove a similar result that depends on cardinalities. In Boolean matrix terminology it states that $R$ can be represented as the union of $k$ pairwise disjoint univalent subrelations if and only if $R$ has at most $k$ 1-entries in a row. If $X$ equals $Y$ and $R$ is the *adjacency relation* of a (finite) directed graph $g = (X, R)$, then $k$ is the maximum outdegree of $g$ and usually much smaller than the number $|X|$ of vertices. In the proof of the following Theorem 6.1 we apply the fact that each relation contains a maximal univalent subrelation with respect to inclusion or, in other words, that the following relational variant of the set-theoretic axiom of choice holds.

**Axiom 6.1** (Relational Axiom of Choice)**.** *For all relations $R$ there exists a univalent relation $F$ such that $F \subseteq R$ and $\mathit{dom}(F) = \mathit{dom}(R)$.*

This variant of the set-theoretic axiom of choice can be found as property **AC 4** in [23], with $\mathcal{D}(S)$ as notation for the set-theoretical domain of a relation $S$ instead of our vector description $\mathit{dom}(S)$. As we only deal with set-theoretic relations, besides the axioms (R1) to (R4) of a relation algebra, the singleton set axiom, the point axiom and the cardinality axioms (C1) to (C5) we are allowed to assume Axiom 6.1. This completes the list of the twelve axioms that are used in the present paper.

The proof of direction "(ii) $\Longrightarrow$ (i)" of the subsequent Theorem 6.1 is by induction on $k$ and, in principle, describes an algorithm for the computation of the set of $k$ univalent subrelations of $R$. Since in (2) a cardinality is used, by Convention 4.1 the carrier sets of $R$ are assumed as finite and, hence, the univalent subrelation $F$ of Axiom 6.1 can easily be computed by a relational program, for instance, formulated in the programming language of the RelView tool.

**Theorem 6.1.** *For all $k \in \mathbb{N}$ and relations $R : X \leftrightarrow Y$ the following facts are equivalent:*

*(i) There exists a set $\{ F_1, \ldots, F_k \}$ of $k$ pairwise disjoint and univalent relations with $R = \bigcup_{i=1}^{k} F_i$.*

*(ii) For all points $p : X \leftrightarrow \mathbb{1}$ it holds $|R^{\mathsf{T}};p| \leq k$.*

*Proof.* "(i) $\Longrightarrow$ (ii)": Let $R : X \leftrightarrow Y$ be an arbitrary relation and $\{ F_1, \ldots, F_k \}$ be a set of $k$ univalent relations with $R = \bigcup_{i=1}^{k} F_i$. Then the following calculation that proves the claim:

$$\begin{aligned}
\left|R^{\mathsf{T}};p\right| &= \left|(\textstyle\bigcup_{i=1}^{k} F_i)^{\mathsf{T}};p\right| && \text{as } R = \textstyle\bigcup_{i=1}^{k} F_i \\
&= \left|\textstyle\bigcup_{i=1}^{k} F_i^{\mathsf{T}};p\right| \\
&\leq \textstyle\sum_{i=1}^{k} \left|F_i^{\mathsf{T}};p\right| && \text{by axiom (C3)} \\
&\leq \textstyle\sum_{i=1}^{k} |p| && \text{Theorem 4.2.(i), since } F_i \text{ is univalent} \\
&= k \cdot |p| \\
&= k && \text{by Lemma 5.1 .}
\end{aligned}$$

"(ii) $\Longrightarrow$ (i)": As we have already mentioned, the proof of this direction is by induction. We show for all $k \in \mathbb{N}$ that for all $R : X \leftrightarrow Y$ if $|R^{\mathsf{T}};p| \leq k$, for all points $p : X \leftrightarrow \mathbb{1}$, then there exists a set $\{ F_1, \ldots, F_k \}$ of $k$ pairwise disjoint and univalent relations such that $R = \bigcup_{i=1}^{k} F_i$.

For the induction base $k = 0$ let $R : X \leftrightarrow Y$ be a relation such that $|R^{\mathsf{T}};p| \leq 0$, for all points $p : X \leftrightarrow \mathbb{1}$. Then axiom (C1) yields $R^{\mathsf{T}};p = \mathsf{O}$, for all points $p : X \leftrightarrow \mathbb{1}$. Due to the equation $\mathsf{O} = \bigcup \emptyset$ (and since all relations from an empty set of relations are pairwise disjoint), it suffices to show $R = \mathsf{O}$. Using Lemma 3.1.(ii) in the second step, this is done by

$$R^{\mathsf{T}} = R^{\mathsf{T}};\mathsf{I} = R^{\mathsf{T}};\textstyle\bigcup_{p \in \mathcal{P}(\mathsf{L}_{X\mathbb{1}})} p;p^{\mathsf{T}} = \textstyle\bigcup_{p \in \mathcal{P}(\mathsf{L}_{X\mathbb{1}})} R^{\mathsf{T}};p;p^{\mathsf{T}} = \mathsf{O} .$$

For the induction step, let $k \neq 0$ and assume again an arbitrary $R : X \leftrightarrow Y$ such that $|R^{\mathsf{T}};p| \leq k$, for all points $p : X \leftrightarrow \mathbb{1}$. Due to Axiom 6.1 there exists $F : X \leftrightarrow Y$ such that $F^{\mathsf{T}};F \subseteq \mathsf{I}$, $F \subseteq R$ and $\mathit{dom}(F) = \mathit{dom}(R)$.

We want to apply the induction hypothesis to $R \cap \overline{F}$. So, we first have to show show $|(R \cap \overline{F})^{\mathsf{T}};p| \leq k - 1$, for all points $p : X \leftrightarrow \mathbb{1}$. For the proof let $p : X \leftrightarrow \mathbb{1}$ be an arbitrary point. We consider two cases. If $|R^{\mathsf{T}};p| \leq k - 1$, then the monotonicity of the cardinality operation yields

$$\left|(R \cap \overline{F})^{\mathsf{T}};p\right| \leq \left|R^{\mathsf{T}};p\right| \leq k - 1 .$$

Now let $|R^{\mathsf{T}};p| = k$. Since $k \neq 0$ we get $R^{\mathsf{T}};p \neq \mathsf{O}$ with axiom (C1) and, moreover, $p \subseteq \mathit{dom}(R)$ with Lemma 3.3. From $\mathit{dom}(F) = \mathit{dom}(R)$ we obtain $p \subseteq \mathit{dom}(F)$. Furthermore, $F \subseteq R$ implies $F^{\mathsf{T}};p \subseteq R^{\mathsf{T}};p$. Next, we verify the auxiliary fact $R^{\mathsf{T}};p \cap \overline{F}^{\mathsf{T}};p \subset R^{\mathsf{T}};p$. It suffices to show that $R^{\mathsf{T}};p \cap \overline{F}^{\mathsf{T}};p \neq R^{\mathsf{T}};p$ holds. We have $F^{\mathsf{T}};p \subseteq R^{\mathsf{T}};p$, which by Boolean algebra rules is equivalent to $F^{\mathsf{T}};p \cap \overline{R^{\mathsf{T}};p} = \mathsf{O}$. Since $p \subseteq \mathit{dom}(F) = F;\mathsf{L}$, Lemma 3.3 yields $F^{\mathsf{T}};p \neq \mathsf{O}$. Thus we find

$$\mathsf{O} \neq F^{\mathsf{T}};p = F^{\mathsf{T}};p \cap \mathsf{L} = F^{\mathsf{T}};p \cap \left(R^{\mathsf{T}};p \cup \overline{R^{\mathsf{T}};p}\right) = \left(F^{\mathsf{T}};p \cap R^{\mathsf{T}};p\right) \cup \left(F^{\mathsf{T}};p \cap \overline{R^{\mathsf{T}};p}\right) = F^{\mathsf{T}};p \cap R^{\mathsf{T}};p\ ,$$

The above is equivalent to $R^{\mathsf{T}};p \not\subseteq \overline{F^{\mathsf{T}};p}$. Since $p$ is a point, we have $\overline{F^{\mathsf{T}};p} = \overline{F^{\mathsf{T}}};p = \overline{F}^{\mathsf{T}};p$, which yields:

$$R^{\mathsf{T}};p \neq R^{\mathsf{T}};p \cap \overline{F^{\mathsf{T}};p} = R^{\mathsf{T}};p \cap \overline{F}^{\mathsf{T}};p\ .$$

Using the auxiliary fact and the strict monotonicity of the cardinality operation we get $|(R \cap \overline{F})^{\mathsf{T}};p| \leq k-1$ from

$$|(R \cap \overline{F})^{\mathsf{T}};p| \leq |R^{\mathsf{T}};p \cap \overline{F}^{\mathsf{T}};p| < |R^{\mathsf{T}};p| = k\ .$$

The just shown result allows to apply the induction hypothesis to $R \cap \overline{F}$. Hence, there exists a set $\{ F_1, \ldots, F_{k-1} \}$ of $k-1$ pairwise disjoint univalent relations such that $R \cap \overline{F} = \bigcup_{i=1}^{k-1} F_i$. Since $F$ is univalent and

$$R = F \cup (R \cap \overline{F}) = F \cup \textstyle\bigcup_{i=1}^{k-1} F_i\ ,$$

we get $R$ as union of the $k$ univalent relations of the set $\{ F, F_1, \ldots, F_{k-1} \}$. The fact that these relations are pairwise disjoint follows from $F \cap F_i \subseteq F \cap R \cap \overline{F} = \mathsf{O}$, for all $i \in \{ 1, \ldots, k-1 \}$, and the induction hypothesis, which states that the elements of the set $\{ F_i \mid i \in \{ 1, \ldots, k-1 \} \}$ are pairwise disjoint. □

By transposing the relation in question we get from this theorem that $R : X \leftrightarrow Y$ is the disjoint union of $k$ injective relations if and only if $|R;p| \leq k$, for all points $p : Y \leftrightarrow \mathbb{1}$, i.e., if and only if in Boolean matrix terminology $R$ has at most $k$ 1-entries in a column.

In the remainder of the section we investigate cardinality properties with respect to univalence and totality, and, by combining them, with respect to the mapping property. By transposing the relation in question and using axiom (C2) the results we will prove immediately lead to cardinality properties with respect to injectivity and surjectivity and, hence, to bijective mappings in the relational sense. We start with the following characterisation of univalent relations.

**Theorem 6.2.** *For all relations $R : X \leftrightarrow Y$ the following facts are equivalent:*

*(i)* *$R$ is univalent.*

*(ii)* *For all points $p : X \leftrightarrow \mathbb{1}$ it holds $\left|R^{\mathsf{T}};p\right| \leq 1$.*

*(iii)* $|\mathit{dom}(R)| = |R|$.

*Proof.* "(i) $\Longrightarrow$ (ii)": Suppose that $R$ is univalent. Let $p : X \leftrightarrow \mathbb{1}$ to be a point. Since $R$ is univalent, Theorem 4.2.(i) and Lemma 5.1 yield $\left|R^{\mathsf{T}};p\right| \leq |p| = 1$.

"(ii) $\Longrightarrow$ (i)": This direction immediately follows from the same direction of Theorem 6.1, taking $k$ as 1.

"(ii) $\Longrightarrow$ (iii)": Because of property (6) we have the estimate $|\mathit{dom}(R)| \leq |R|$ and it suffices to show $|R| \leq |\mathit{dom}(R)|$. To reach this goal, we calculate as follows:

$$\begin{array}{rll} |R| & = \sum_{p \in \mathcal{P}(\mathsf{L}_{X\mathbb{1}})} \left|R^{\mathsf{T}};p\right| & \text{row-variant of Theorem 4.1} \\ & = \sum_{p \in \mathcal{P}(\mathit{dom}(R))} \left|R^{\mathsf{T}};p\right| & \text{see below} \\ & \leq \sum_{p \in \mathcal{P}(\mathit{dom}(R))} 1 & \text{assumption} \\ & = |\mathcal{P}(\mathit{dom}(R))| & \\ & = |\mathit{dom}(R)| & \text{by Lemma 5.2}\ . \end{array}$$

Here the second step uses that for all points $p : X \leftrightarrow \mathbb{1}$ from $p \notin \mathcal{P}(dom(R))$ it follows $\left|R^{\mathsf{T}};p\right| = 0$ because of Lemma 3.3 and axiom (C1).

"(iii) $\Longrightarrow$ (ii)": We start with the calculation

$$|\mathcal{P}(dom(R))| = |dom(R)| = |R| = \sum_{p \in \mathcal{P}(dom(R))} \left|R^{\mathsf{T}};p\right| ,$$

where we use Lemma 5.2, then the assumption and, finally, the first two steps of the above calculation. Lemma 3.3 and axiom (C1) imply $\left|R^{\mathsf{T}};p\right| \neq 0$, for all $p \in \mathcal{P}(dom(R))$ and so the above equation yields $\left|R^{\mathsf{T}};p\right| = 1$, for all $p \in \mathcal{P}(dom(R))$. For all points $p : X \leftrightarrow \mathbb{1}$ with $p \notin \mathcal{P}(dom(R))$ the inclusion $p \subseteq dom(R)$ is false and in this case Lemma 3.3 and axiom (C1) yield $\left|R^{\mathsf{T}};p\right| = 0 \leq 1$. □

In case of total relations we have the following result. In contrast with Theorem 6.2 it seems not to exist a characterisation of the totality of a relation $R$ by relating $|dom(R)|$ and $|R|$ in a simple manner. But there is one if $|dom(R)|$ is compared with the cardinality of the source of $R$.

**Theorem 6.3.** *For all relations $R : X \leftrightarrow Y$ the following facts are equivalent:*

*(i)* *$R$ is total.*

*(ii)* *For all points $p : X \leftrightarrow \mathbb{1}$ it holds $\left|R^{\mathsf{T}};p\right| \geq 1$.*

*(iii)* $|dom(R)| = |X|$.

*Proof.* "(i) $\Longrightarrow$ (ii)": Let $p : X \leftrightarrow \mathbb{1}$ be a point. Then we obtain

$$\left|R^{\mathsf{T}};p\right| = \left|p^{\mathsf{T}};R\right| = \left|p^{\mathsf{T}};R \cap \mathsf{L}_{\mathbb{1}X}\right| \geq \left|p^{\mathsf{T}} \cap \mathsf{L}_{\mathbb{1}X};R^{\mathsf{T}}\right| = |p \cap R;\mathsf{L}_{X\mathbb{1}}| = |p \cap \mathsf{L}_{X\mathbb{1}}| = |p| = 1 ,$$

where we use Axiom (C2) in the first and fourth step, Axiom (C4) in the third step (since $p^{\mathsf{T}}$ is univalent), the totality of $R$ in the fifth step, and Lemma 5.1 in the last step.

"(ii) $\Longrightarrow$ (i)": For every point $p : X \leftrightarrow \mathbb{1}$ we have the following equivalences, where the last one is due to Lemma 3.3:

$$\textbf{true} \iff \left|R^{\mathsf{T}};p\right| \geq 1 \iff R^{\mathsf{T}};p \neq \mathsf{O} \iff p \subseteq R;\mathsf{L}_{X\mathbb{1}} .$$

We thus get $\mathsf{L}_{X\mathbb{1}} = \bigcup_{p \in \mathcal{P}(\mathsf{L}_{X\mathbb{1}})} p \subseteq R;\mathsf{L}_{X\mathbb{1}} \subseteq \mathsf{L}_{X\mathbb{1}}$ by Axiom 3.1 and a property of the supremum. Thus $R;\mathsf{L}_{X\mathbb{1}} = \mathsf{L}_{X\mathbb{1}}$ and since $R;\mathsf{L}_{XY} = R;\mathsf{L}_{X\mathbb{1}};\mathsf{L}_{\mathbb{1}Y} = \mathsf{L}_{X\mathbb{1}};\mathsf{L}_{\mathbb{1}Y} = \mathsf{L}_{XY}$ by Equation (4), we obtain the totality of $R$.
"(i) $\Longrightarrow$ (iii)": The totality of $R$ implies $R;\mathsf{L}_{Y\mathbb{1}} = \mathsf{L}_{X\mathbb{1}}$ and from this equation we get

$$|dom(R)| = |R;\mathsf{L}_{Y\mathbb{1}}| = |\mathsf{L}_{X\mathbb{1}}| = |X| .$$

"(iii) $\Longrightarrow$ (i)": If $|dom(R)| = |X|$, then we have $|R;\mathsf{L}_{Y\mathbb{1}}| = |\mathsf{L}_{X\mathbb{1}}|$. This equation leads to $R;\mathsf{L}_{Y\mathbb{1}} = \mathsf{L}_{X\mathbb{1}}$, since from the assumption $R;\mathsf{L}_{Y\mathbb{1}} \subset \mathsf{L}_{X\mathbb{1}}$ we would obtain the contradiction $|dom(R)| = |R;\mathsf{L}_{Y\mathbb{1}}| < |\mathsf{L}_{X\mathbb{1}}| = |X|$ because of the strict monotonicity of the cardinality operation. But $R;\mathsf{L}_{Y\mathbb{1}} = \mathsf{L}_{X\mathbb{1}}$ leads to $R;\mathsf{L} = \mathsf{L}$, for all universal relations $\mathsf{L} : Y \leftrightarrow Z$, which is the totality of the relation $R$. □

Theorem 6.2 and Theorem 6.3 show together that a relation $R : X \leftrightarrow Y$ is a mapping if and only if $\left|R^{\mathsf{T}};p\right| = 1$, for all points $p : X \leftrightarrow \mathbb{1}$, or, in Boolean matrix terminology, if and only if each row possesses exactly one 1-entry. Furthermore, Theorem 6.2 and axiom (C2) show that $R$ is univalent and injective (that is, a *matching* in the relational sense; see e.g., [16]) if and only if $|dom(R)| = |R| = |ran(R)|$. This result is a strengthening of Corollary 1(e) of [16].

## 7. Some Simple Applications

Before presenting a more complex application in Section 8, in this section we present some simple examples that use the results we have obtained so far. First, we compute the cardinality of linear order relations in terms of the cardinalities of their carrier sets.

**Theorem 7.1.** *For all linear order relations $R : X \leftrightarrow X$ we have $|R| = \frac{|X|^2+|X|}{2}$.*

*Proof.* The equation immediately follows from the calculation

$$|X|^2 = |\mathsf{L}_{XX}| = \left|R \cup R^\mathsf{T}\right| = |R| + \left|R^\mathsf{T}\right| - |\mathsf{I}_X| = |R| + |R| - |X| \,,$$

that uses Lemma 5.3.(i), axiom (C3) (reflexivity and antisymmetry of $R$ imply $\mathsf{I} = R \cap R^\mathsf{T}$), axiom (C2) and Lemma 5.3.(ii). □

As a motivation for the next application we assume $g = (X, E)$ to be an undirected (loop-free) graph with (finite) vertex set $X$ and edge set $E$, where edges are two-element subsets of $X$. As shown for instance in [27], in many applications the *incidence relation* $R : E \leftrightarrow X$ is adequate to model $g$. It relates an edge $e \in E$ and a vertex $x \in X$ if and only if $x \in e$. In an undirected graph each edge is incident to precisely two vertices, or, in Boolean matrix terminology each row of the incidence relation contains precisely two 1-entries. Via the cardinality operation this can be specified by demanding $|R^\mathsf{T};p| = 2$, for all points $p : E \leftrightarrow \mathbb{1}$. The next theorem presents an equivalent specification. We formulate it for general relations and not for incidence relations of undirected graphs only.

**Theorem 7.2.** *For all relations $R : X \leftrightarrow Y$ the following facts are equivalent:*

*(i)* $\left|R^\mathsf{T};p\right| = 2$*, for all points $p : X \leftrightarrow \mathbb{1}$.*

*(ii) There exist mappings $F, G : X \leftrightarrow Y$ such that $F \cap G = \mathsf{O}$ and $F \cup G = R$.*

*Proof.* "(i) $\Longrightarrow$ (ii)": From Theorem 6.1 we obtain that there exist disjoint univalent relations $F, G : X \leftrightarrow Y$ such that $F \cup G = R$. It remains to show their totality. We want to apply Theorem 6.3 and, therefore, have to verify $\left|F^\mathsf{T};p\right| \geq 1$ and $\left|G^\mathsf{T};p\right| \geq 1$, for all points $p : X \leftrightarrow \mathbb{1}$. So, let $p : X \leftrightarrow \mathbb{1}$ be an arbitrary point. Using the assumption, $F \cup G = R$, the axioms (C1) and (C3), where $F^\mathsf{T};p \cap G^\mathsf{T};p = \mathsf{O}$ is again a consequence of $F \cap G = \mathsf{O}$, we calculate as follows:

$$2 = \left|R^\mathsf{T};p\right| = \left|(F \cup G)^\mathsf{T};p\right| = \left|F^\mathsf{T};p \cup G^\mathsf{T};p\right| = \left|F^\mathsf{T};p\right| + \left|G^\mathsf{T};p\right| \,. \tag{7}$$

Since $F$ and $G$ are univalent, Theorem 6.2 shows $\left|F^\mathsf{T};p\right| \leq 1$ and $\left|G^\mathsf{T};p\right| \leq 1$ and the Equation (7) yields $\left|F^\mathsf{T};p\right| = 1$ and $\left|G^\mathsf{T};p\right| = 1$.

"(ii) $\Longrightarrow$ (i)": Suppose that $F$ and $G$ are mappings with the required properties and let $p : X \leftrightarrow \mathbb{1}$ to be a point. Since $p$ is injective and $F \cap G = \mathsf{O}$ we get

$$p^\mathsf{T};F \cap p^\mathsf{T};G = p^\mathsf{T};(F \cap G) = \mathsf{O}$$

by Equation (3). This allows to conclude the proof as follows:

$$\begin{aligned}
\left|R^\mathsf{T};p\right| &= \left|p^\mathsf{T};R\right| && \text{by axiom (C2)} \\
&= \left|p^\mathsf{T};(F \cup G)\right| && \text{assumption} \\
&= \left|p^\mathsf{T};F \cup p^\mathsf{T};G\right| && \\
&= \left|p^\mathsf{T};F\right| + \left|p^\mathsf{T};G\right| && \text{by axioms (C1), (C3) and above result} \\
&= \left|p^\mathsf{T}\right| + \left|p^\mathsf{T}\right| && \text{by Lemma 4.1.(ii) ($p^\mathsf{T}$ univalent, $F, G$ mappings)} \\
&= 2 \cdot |p| && \text{by axiom (C2)} \\
&= 2 && \text{by Lemma 5.1 .}
\end{aligned}$$
□

The proof of Proposition 9.1.6 of [27] is an example of an application of the fact that the incidence relation of an undirected graph is the union of two mappings. Other proofs of this textbook, which deal with undirected graphs $g = (X, E)$ allowing loops, i.e., the restriction $|e| = 2$, for all $e \in E$, is weakened to $1 \leq |e| \leq 2$, use that the incidence relation of $g$ is the union of two univalent relations. Examples are the proofs of Proposition 5.4.5 and Proposition 9.2.2. However, as in the case of cardinalities, in [27] these facts are treated rather intuitively and not formally proved as we do in Theorem 7.2 and Theorem 6.1, respectively.

Theorem 7.2 can be generalised as follows: For all relations $R : X \leftrightarrow Y$ and all $k \in \mathbb{N}$ we have that $|R^\mathsf{T};p| = k$, for all points $p : X \leftrightarrow \mathbb{1}$, if and only if there exists a set $\{ F_1, \ldots, F_k \}$ of $k$ pairwise disjoint mappings such that

$R = \bigcup_{i=1}^{k} F_i$. For Theorem 7.2 and its generalisation there are also "column-oriented" versions with injective and surjective relations instead of mappings.

As continuation of [16], in [17] Y. Kawahara and M. Winter investigate the cardinality of relations in the context of allegories. In the introduction of [17] they mention as a possible application the characterisation of finite trees as those connected undirected graphs for which $m = n - 1$ holds, with $m$ as the number of edges and $n$ as the number of vertices. To keep the corresponding equation for relations, we have to consider adjacency relations of directed trees $g = (X, R)$. In [27] such relations $R$ are specified by injectivity $R;R^{\mathsf{T}} \subseteq \mathsf{I}$, cycle-freeness $R^+ \subseteq \overline{\mathsf{I}}$ and the existence of a *root*, i.e., of a point $r$ such that $r;\mathsf{L} \subseteq R^*$ (or, in words, such that each vertex of $g$ is reachable from the vertex modelled by $r$). If we call a relation with these properties a *rooted tree relation* with root $r$, then we are able to prove the desired equation and also that it characterises the rooted tree relations among the relations that possess a root. Note, that the latter fact corresponds to the notion "connected" in case of undirected graphs. The proof of the following lemma and parts of the proof of Theorem 7.3 stem from W. Guttmann and have been developed while discussing the cardinality of relations with the first author during the RAMiCS 2015 conference.

**Lemma 7.1.** *For all injective relations $R$ we have $R^*;(R^+ \cap \mathsf{I});\mathsf{L} \subseteq R^{\mathsf{T}};\mathsf{L}$.*

*Proof.* First, we apply induction to prove $R^k;(R^+ \cap \mathsf{I});\mathsf{L} \subseteq (R^* \cap R^{\mathsf{T}});\mathsf{L}$, for all $k \in \mathbb{N}$. The induction base $k = 0$ is shown as follows:

$$
\begin{aligned}
R^0;(R^+ \cap \mathsf{I});\mathsf{L} &= \mathsf{I};(R^+ \cap \mathsf{I});\mathsf{L} && \text{definition powers}\\
&= (R^*;R \cap \mathsf{I});\mathsf{L} && \text{property of closures}\\
&\subseteq (R^* \cap \mathsf{I};R^{\mathsf{T}});(R \cap (R^*)^{\mathsf{T}};\mathsf{I});\mathsf{L} && \text{Dedekind rule (1)}\\
&\subseteq (R^* \cap R^{\mathsf{T}});\mathsf{L}\,.
\end{aligned}
$$

For the induction step, let $k \neq 0$. Then we get:

$$
\begin{aligned}
R^k;(R^+ \cap \mathsf{I});\mathsf{L} &= R;R^{k-1};(R^+ \cap \mathsf{I});\mathsf{L} && \text{definition powers}\\
&\subseteq R;(R^* \cap R^{\mathsf{T}});\mathsf{L} && \text{by induction hypothesis}\\
&\subseteq (R;R^* \cap R;R^{\mathsf{T}});\mathsf{L} &&\\
&\subseteq (R^*;R \cap \mathsf{I});\mathsf{L} && R \text{ injective}, R;R^* = R^*;R\\
&\subseteq (R^* \cap R^{\mathsf{T}});\mathsf{L} && \text{see induction base}\,.
\end{aligned}
$$

Now the following calculation yields the desired result:

$$
\begin{aligned}
R^*;(R^+ \cap \mathsf{I});\mathsf{L} &= (\textstyle\bigcup_{k\geq 0} R^k);(R^+ \cap \mathsf{I});\mathsf{L} &&\\
&= \textstyle\bigcup_{k\geq 0} R^k;(R^+ \cap \mathsf{I});\mathsf{L} &&\\
&\subseteq (R^* \cap R^{\mathsf{T}});\mathsf{L} && \text{above result}\\
&\subseteq R^{\mathsf{T}};\mathsf{L}\,.
\end{aligned}
$$

□

If $R : X \leftrightarrow X$ is the adjacency relation of a directed graph $g = (X, R)$, then the vector $R^*;(R^+ \cap \mathsf{I});\mathsf{L} : X \leftrightarrow \mathbb{1}$ models the set of vertices of $g$ from which a cycle of $g$ can be reached. So, in graph-theoretic terminology the lemma says that in a directed graph with no backward-branchings each such vertex has a predecessor, i.e., lies on the reachable cycle. After this preparation we now can show the desired characterisation of the rooted tree relations by means of their cardinalities. Note, that in the following theorem the point property of $r$ implies $r \neq \mathsf{O}$ which, in turn, implies $X \neq \emptyset$ such that $|X| - 1$ is defined.

**Theorem 7.3.** *For all relations $R : X \leftrightarrow X$ with root $r : X \leftrightarrow \mathbb{1}$ the following facts are equivalent:*

*(i)* *$R$ is injective and cycle-free (i.e., a rooted tree relation).*

*(ii)* $|R| = |X| - 1$.

*Proof.* We prepare the proof of the claimed equivalence by first showing that the existence of the root $r$ implies $|R;p| \geq 1$, for all $p \in \mathcal{P}(\overline{r})$. Due to axiom (C1) it suffices to show $R;p \neq \mathsf{O}$, for all $p \in \mathcal{P}(\overline{r})$. We use contraposition and show that for all points $p \in \mathcal{P}(\mathsf{L}_{X\mathbb{1}})$ the equality $R;p = \mathsf{O}$ implies that $p = r$ and thus $p \notin \mathcal{P}(\overline{r})$, because $p$ is a point.

Let $p \in \mathcal{P}(\mathsf{L}_{X\mathbb{1}})$ such that $R;p = \mathsf{O}$ holds. Then we calculate $R^+;p = (\bigcup_{k>0} R^k);p = \bigcup_{k>0} R^k;p = \mathsf{O}$ (using induction). Since $r;\mathsf{L} \subseteq R^*$, we have that $r;p^\mathsf{T} \subseteq r;\mathsf{L} \subseteq R^*$, which is equivalent to $r \subseteq R^*;p$, because $r, p$ are points. Now we get

$$r \subseteq R^*;p = (R^+ \cup \mathsf{I});p = R^+;p \cup \mathsf{I};p = p$$

which, in turn, yields $r = p$ as $p$ and $r$ are points.

Having shown $|R;p| \geq 1$, for all $p \in \mathcal{P}(\overline{r})$, now we start the proof of the equivalence of (i) and (ii).

"(i) $\Longrightarrow$ (ii)": We first show that $R;r = \mathsf{O}$ holds. Using the Dedekind rule in the first, the root property in the third, and the injectivity of $r$ in the second to last step, we obtain

$$R^\mathsf{T};\mathsf{L} \cap r = \left(R^\mathsf{T} \cap r;\mathsf{L}\right);(\mathsf{L} \cap R;r) = \left(R^\mathsf{T} \cap r;\mathsf{L}\right);R;r \subseteq \left(R^\mathsf{T} \cap R^*\right);R;r \subseteq R^*;R;r = R^+;r \subseteq \overline{\mathsf{I}};r \subseteq \overline{\mathsf{I};r} = \overline{r}\ .$$

We also have $R^\mathsf{T};\mathsf{L} \cap r \subseteq r$ and thus $R^\mathsf{T};\mathsf{L} \cap r \subseteq \overline{r} \cap r = \mathsf{O}$. By Boolean algebra rules we get that $R^\mathsf{T};\mathsf{L} \subseteq \overline{r}$, which by a Schröder equivalence is equivalent to $R;r \subseteq \mathsf{O}$. Thus $R;r = \mathsf{O}$ holds. As a consequence, axiom (C1) yields $|R;r| = 0$. Next, we prove $|R;p| \leq 1$ for all $p \in \mathcal{P}(\overline{r})$. Let $p \in \mathcal{P}(\overline{r})$. Since $R$ is injective, $R^\mathsf{T}$ is univalent and thus Theorem 4.2.(i) and Lemma 5.1 imply

$$|R;p| = \left|R^{\mathsf{T}^\mathsf{T}};p\right| \leq |p| = 1\ .$$

Together with the result at the beginning of the proof this implies $|R;p| = 1$, for all $p \in \mathcal{P}(\overline{r})$. Now the following calculation shows equation (ii):

$$\begin{aligned}
|R| &= \textstyle\sum_{p\in\mathcal{P}(\mathsf{L}_{X\mathbb{1}})} |R;p| && \text{by Theorem 4.1}\\
&= |R;r| + \textstyle\sum_{p\in\mathcal{P}(\overline{r})} |R;p| && \text{as } \mathcal{P}(\mathsf{L}_{X\mathbb{1}}) = \{\, r\,\} \cup \mathcal{P}(\overline{r})\\
&= \textstyle\sum_{p\in\mathcal{P}(\overline{r})} 1 && \text{auxiliary results}\\
&= |\mathcal{P}(\overline{r})|\\
&= |\overline{r}| && \text{by Lemma 5.2}\\
&= |X| - 1 && \text{see Section 5}\ .
\end{aligned}$$

"(ii) $\Longrightarrow$ (i)": First, we verify that $R$ is an injective relation. To this end, we start with the equation

$$|\mathcal{P}(\overline{r})| = |\overline{r}| = |X| - 1 = |R| = \textstyle\sum_{p\in\mathcal{P}(\mathsf{L}_{X\mathbb{1}})} |R;p| = |R;r| + \sum_{p\in\mathcal{P}(\overline{r})} |R;p|$$

where we use Lemma 5.2, equation (ii) and Theorem 4.1. At the beginning of the proof we have shown $|R;p| \geq 1$, for all $p \in \mathcal{P}(\overline{r})$. Hence the equation and Lemma 5.2 yield $|R;p| = 1$, for all $p \in \mathcal{P}(\overline{r})$, and $|R;r| = \mathsf{O}$. So, Theorem 6.2 implies that $R^\mathsf{T}$ is univalent, i.e., $R$ is injective.

It remains to verify that $R$ is cycle-free. Above we have shown that $|R;r| = 0$ holds, which by (C1) yields $R;r = \mathsf{O}$. Using the point property $r^\mathsf{T};r = \mathsf{L}_{\mathbb{1}\mathbb{1}} = \mathsf{I}_\mathbb{1}$ in the first, the root property of $r$ in the second, Lemma 7.1 in the third, $R;r = \mathsf{O}$ (as well as $\mathsf{O}^\mathsf{T} = \mathsf{O}$ and $\mathsf{O};S = \mathsf{O}$) in the second to last, and Axiom 2.2 in the last step we get

$$\mathsf{L}_{\mathbb{1}X};(R^+ \cap \mathsf{I});\mathsf{L}_{X\mathbb{1}} = r^\mathsf{T};r;\mathsf{L}_{\mathbb{1}X};(R^+ \cap \mathsf{I});\mathsf{L}_{X\mathbb{1}} \subseteq r^\mathsf{T};R^*;(R^+ \cap \mathsf{I});\mathsf{L}_{X\mathbb{1}} \subseteq r^\mathsf{T};R^\mathsf{T};\mathsf{L}_{X\mathbb{1}} = (R;r)^\mathsf{T};\mathsf{L}_{X\mathbb{1}} = \mathsf{O}_{\mathbb{1}\mathbb{1}} \neq \mathsf{L}_{\mathbb{1}\mathbb{1}}\ .$$

The Tarski rule implies that $R^+ \cap \mathsf{I} = \mathsf{O}$ holds, which by Boolean algebra rules is equivalent to $R^+ \subseteq \overline{\mathsf{I}}$. □

In the remainder of this section we apply the hitherto results for proving some estimates for relational variants of certain graph parameters. The original graph-theoretic versions are well-known, see e.g., [12]. In the context of relations some of the following results are presented in [27]. However, as already mentioned in the introduction, in [27] the cardinality of relations is treated rather informally and intuitively only. Therefore, the proofs of [27] are not given in the purely calculational and algebraic way as we will do in the following.

We start with the stability number, in the context of undirected graphs also known as independence number. Assume $g = (X, R)$ to be a directed graph and $S \subseteq X$. Then $S$ is stable in $g$ if $(x, y) \notin R$, for all $x, y \in S$. A little point-wise reasoning shows that a vector $s : X \leftrightarrow \mathbb{1}$ models a stable set of $g$ if and only if $R;s \subseteq \overline{s}$. Therefore, following [27] we call for a given relation $R : X \leftrightarrow X$ a vector $s : X \leftrightarrow \mathbb{1}$ *R-stable* if $R;s \subseteq \overline{s}$. Furthermore, by

$$\alpha_R := \max\{\, |s| \mid s \text{ is } R\text{-stable}\,\}$$

we define the *stability number* of $R$. The next theorem presents an upper bound for this number in terms of the relational cardinality operation.

**Theorem 7.4.** *For all relations $R : X \leftrightarrow X$ and $R$-stable vectors $s : X \leftrightarrow \mathbb{1}$ we have $|s| \leq \sqrt{|\overline{R}|}$, hence $\alpha_R \leq \sqrt{|\overline{R}|}$.*

*Proof.* The first claim follows from the subsequent calculation:

$$\begin{aligned} R;s \subseteq \overline{s} &\iff s;s^{\mathsf{T}} \subseteq \overline{R} && \text{by a Schröder rule} \\ &\implies \left|s;s^{\mathsf{T}}\right| \leq \left|\overline{R}\right| && \text{monotonicity of cardinality operation} \\ &\iff |s|^2 \leq \left|\overline{R}\right| && \text{by Theorem 5.1} \\ &\iff |s| \leq \sqrt{\left|\overline{R}\right|}\,. \end{aligned}$$

The second claim is an immediate consequence of the first one by taking an $R$-stable vector $s : X \leftrightarrow \mathbb{1}$ with $|s| = \alpha_R$, i.e., a maximum one. □

For symmetric and irreflexive relations, especially for adjacency relations of undirected graphs, it is also possible to prove a lower bound for the stability number $\alpha_R$ that depends on the graph's vertex number and maximum degree.

**Theorem 7.5.** *For all symmetric and irreflexive relations $R : X \leftrightarrow X$ we have $\frac{|X|}{\Delta_R+1} \leq \alpha_R$, where the natural number $\Delta_R$ is defined by $\Delta_R := \max\{\ |R;p| \mid p \in \mathcal{P}(\mathsf{L}_{X\mathbb{1}})\ \}$.*

*Proof.* We take an $R$-stable vector $s : X \leftrightarrow \mathbb{1}$ such that $|s| = \alpha_R$, i.e., a maximum one. Then $s$ is also a maximal $R$-stable vector, since the existence of an $R$-stable vector $t : X \leftrightarrow \mathbb{1}$ with $s \subset t$ would lead to the contradiction $\alpha_R = |s| < |t| \leq \alpha_R$ (because of the strict monotonicity of the cardinality operation).

In [27], Proposition 8.1.3, it is shown that for a symmetric and irreflexive relation $R$ the maximality of $s$ leads to $R;s = \overline{s}$. The desired result is now an immediate consequence of the following estimate:

$$\begin{aligned} |X| &= |\mathsf{L}_{X\mathbb{1}}| \\ &= |s \cup \overline{s}| \\ &= |s| + |\overline{s}| && \text{by axioms (C1), (C3)} \\ &= |s| + |R;s| && \text{see above} \\ &= |s| + \left|R;\textstyle\bigcup_{p\in\mathcal{P}(s)} p\right| && \text{by Lemma 3.1.(i)} \\ &= |s| + \left|\textstyle\bigcup_{p\in\mathcal{P}(s)} R;p\right| \\ &\leq |s| + \textstyle\sum_{p\in\mathcal{P}(s)} |R;p| && \text{by axiom (C3)} \\ &\leq |s| + \textstyle\sum_{p\in\mathcal{P}(s)} \Delta_R && \text{definition } \Delta_R \\ &= |s| + |\mathcal{P}(s)|\cdot\Delta_R \\ &= |s| + |s|\cdot\Delta_R && \text{by Lemma 5.2} \\ &= \alpha_R\cdot(\Delta_R + 1) && \text{as } |s| = \alpha_R\,. \end{aligned}$$

□

If $g = (X, R)$ is a directed graph, then for each vector $v : X \leftrightarrow \mathbb{1}$ we have that $v$ models a *kernel* of $g$ (i.e., a set $K$ of vertices that is stable and from each vertex outside of $K$ there leads an edge into $K$) if and only if $R;v = \overline{v}$; see [27]. So, from the proof of Theorem 7.5 we get that $\frac{|X|}{\Delta_R+1}$ is a lower bound for the cardinality of each kernel of $g$, where $\Delta_R$ is the maximum in-degree of $g$.

Now we assume $g = (X, E)$ to be an undirected graph. We model $g$ by the symmetric and irreflexive adjacency relation $R : X \leftrightarrow X$, such that $(x, y) \in R$ if and only if $\{x, y\} \in E$, for all $x, y \in X$. Then a (vertex) colouring of $g$ is a function (in the usual mathematical sense) $F : X \to \mathbb{N}$ such that $F(x) = F(y)$ implies $(x, y) \notin R$, for all $x, y \in X$. The latter condition is called the *colouring property*. Colourings also can be defined for directed graphs, provided they are loop-free. Based on this observation, we define for a given irreflexive relation $R : X \leftrightarrow X$ a relation $F : X \leftrightarrow X$ to be an *R-colouring* if $F$ is a mapping in the relational sense of Section 2 and $R;F \subseteq \overline{F}$. The inclusion $R;F \subseteq \overline{F}$ is the relation-algebraic specification of the colouring property. From $R \subseteq \overline{\mathsf{I}}$ we get that the *chromatic number*

$$\chi_R := \min\{\,|ran(F)| \mid F : X \leftrightarrow X \text{ is an } R\text{-colouring}\,\}$$

of $R$ is well-defined, since the identity relation $\mathsf{I} : X \leftrightarrow X$ is an *R-colouring*. At this place it should be remarked that we use vertices as colours. This model avoids set-theoretic anomalies and allows purely relational reasoning. Instead of the powerset $2^X$ of $X$ each powerset can be taken as universe where the targets of colourings have to be from, provided it is large enough to ensure the existence of at least one colouring.

The connection between the stability number $\alpha_R$ and the chromatic number $\chi_R$, that is shown in the following theorem, is again the relational version of a well-known estimate in graph theory.

**Theorem 7.6.** *For all relations $R : X \leftrightarrow X$ and all total relations $F : X \leftrightarrow X$ with the colouring property we have $|X| \leq |ran(F)| \cdot \alpha_R$. In particular, for all irreflexive $R : X \leftrightarrow X$ we have $|X| \leq \chi_R \cdot \alpha_R$.*

*Proof.* Let $R, F : X \leftrightarrow X$ such that $F$ is total. Applying (2) to $F^\mathsf{T}$ and using the definition of $ran(F)$ we obtain

$$\mathsf{O} = F;\overline{F^\mathsf{T};\mathsf{L}_{X\mathbb{1}}} = F;\overline{ran(F)} \tag{8}$$

To show the first claim, we calculate as follows:

$$\begin{aligned}
|X| &= |\mathsf{L}_{X\mathbb{1}}| \\
&= |F;\mathsf{L}_{X\mathbb{1}}| && F \text{ is total} \\
&= \left|F;\left(ran(F) \cup \overline{ran(F)}\right)\right| \\
&= \left|F;ran(F) \cup F;\overline{ran(F)}\right| \\
&= |F;ran(F)| && \text{above auxiliary result (8)} \\
&= \left|F;\left(\textstyle\bigcup_{p \in \mathcal{P}(ran(F))} p\right)\right| && \text{by Axiom 3.1} \\
&= \left|\textstyle\bigcup_{p \in \mathcal{P}(ran(F))} F;p\right| \\
&\leq \textstyle\sum_{p \in \mathcal{P}(ran(F))} |F;p| && \text{by axiom (C3)} \\
&\leq \textstyle\sum_{p \in \mathcal{P}(ran(F))} \alpha_R && \text{as } R;F;p \subseteq \overline{F};p = \overline{F;p}, \text{definition } \alpha_R \\
&= |\mathcal{P}(ran(F))| \cdot \alpha_R \\
&= |ran(F)| \cdot \alpha_R && \text{by Lemma 5.2 .}
\end{aligned}$$

Now suppose that $R$ is irreflexive. Then there exists an $R$-colouring $G : X \leftrightarrow X$ such that $|ran(G)| = \chi_R$, i.e. a colouring with a minimum number of colours. We thus get $|X| \leq |ran(G)| \cdot \alpha_R = \chi_R \cdot \alpha_R$, which proves the second claim. □

A vertex cover of a directed graph $g = (X, R)$ is a subset $C$ of the set $X$ of vertices such that $(x, y) \in R$ implies $x \in C$ or $y \in C$, for all $x, y \in X$. Again a little point-wise reasoning shows that a vector $c : X \leftrightarrow \mathbb{1}$ models a vertex cover of $g$ if and only if $R;\overline{c} \subseteq c$. As in [27] we call, for a given relation $R : X \leftrightarrow X$ a vector $c : X \leftrightarrow \mathbb{1}$ an *$R$-vertex-cover* if $R;\overline{c} \subseteq c$ and define the *vertex cover number* of $R$ as

$$\tau_R := \min\{\, |c| \mid c \text{ is an } R\text{-vertex-cover}\,\} \ .$$

From the above definitions we immediately get that for a given relation $R : X \leftrightarrow X$ a vector $v : X \leftrightarrow \mathbb{1}$ is an $R$-vertex-cover if and only if its complement $\overline{v}$ is $R$-stable. This allows to prove the following connection between the stability number $\alpha_R$ and the vertex cover number $\tau_R$.

**Theorem 7.7.** *For all relations $R : X \leftrightarrow X$ we have $\alpha_R + \tau_R = |X|$.*

*Proof.* Let $c : X \leftrightarrow \mathbb{1}$ be a minimum $R$-vertex-cover. We claim that $\overline{c}$ is a maximum $R$-stable vector. We have already mentioned that it is $R$-stable. To show that it is maximum $R$-stable, let $s : X \leftrightarrow \mathbb{1}$ be an arbitrary $R$-stable vector. Then $\overline{s}$ is an $R$-vertex-cover. Since $c$ is a minimum $R$-vertex-cover, we have $|c| \leq |\overline{s}|$. This yields

$$|s| = |X| - |\overline{s}| \leq |X| - |c| = |\overline{c}| \ ,$$

which shows that $\overline{c}$ is a maximum $R$-stable vector. Thus we have $|c| = \tau_R$ and $|\overline{c}| = \alpha_R$, which immediately yields

$$|X| = |\mathsf{L}_{X\mathbb{1}}| = |c \cup \overline{c}| = |c| + |\overline{c}| = \tau_R + \alpha_R$$

due to axioms (C1) and (C3). □

In Section 6 we have introduced relational matchings as univalent and injective relations. For a given relation $R : X \leftrightarrow Y$ we call a matching $S : X \leftrightarrow Y$ an *R-matching* if $S \subseteq R$ and define the *matching number* of $R$ as

$$\nu_R := \max\{\, |S| \mid S \text{ is an } R\text{-matching} \,\} \ .$$

Relational matchings may be heterogeneous. To be able to compare their cardinalities with the numbers we have introduced so far, we have to assume that they are included in homogeneous relations. In such cases there are simple connections between the cardinalities of vertex covers and matchings, which we show next.

**Theorem 7.8.** *For all relations $R : X \leftrightarrow X$, R-vertex-covers $c : X \leftrightarrow \mathbb{1}$ and R-matchings $S : X \leftrightarrow X$ we have $|S| \leq 2{\cdot}|c|$, hence $\nu_R \leq 2{\cdot}\tau_R$.*

*Proof.* The following calculation shows the first claim:

$$\begin{aligned}
|S| &= |S;(c \cup \overline{c})| && \text{by Lemma 4.1.(ii) ($S$ univalent, $c \cup \overline{c}$ mapping)}\\
&= |S;c \cup S;\overline{c}| && \\
&\leq |S;c| + |S;\overline{c}| && \text{by axiom (C3)}\\
&\leq |S;c| + |R;\overline{c}| && \text{as } S \subseteq R, \text{monotonicity of cardinality operation}\\
&\leq |S;c| + |c| && \text{as } R;\overline{c} \subseteq c, \text{monotonicity of cardinality operation}\\
&\leq |c| + |c| && S;c = S^{\mathsf{T}\mathsf{T}};c,\ S^{\mathsf{T}} \text{ is univalent and Theorem 4.2.(i)}\ .
\end{aligned}$$

Again, the second claim is an immediate consequence of the first one by taking an $R$-vertex-cover $c : X \leftrightarrow \mathbb{1}$ with $|c| = \tau_R$ (a minimum one) and an $R$-matching $S : X \leftrightarrow X$ with $|S| = \nu_R$ (a maximum one). □

Having estimated the matching number $\nu_R$ by the vertex cover number $\tau_R$, we now show how to estimate the vertex cover number by the matching number. Here the maximality of matchings with respect to inclusion plays a decisive role.

**Theorem 7.9.** *For all relations $R : X \leftrightarrow X$ and maximal R-matchings $S : X \leftrightarrow X$ the vector $(S \cup S^{\mathsf{T}});\mathsf{L} : X \leftrightarrow \mathbb{1}$ is an R-vertex-cover and $\tau_R \leq 2{\cdot}\nu_R$.*

*Proof.* Let $R, S : X \leftrightarrow X$ such that $S$ is an $R$-matching. We use contraposition and show that if $(S \cup S^{\mathsf{T}});\mathsf{L}$ is not a vertex cover, then $S$ is not a maximal $R$-matching. Suppose that $R;\overline{(S \cup S^{\mathsf{T}});\mathsf{L}} \cap \overline{(S \cup S^{\mathsf{T}});\mathsf{L}} \neq \mathsf{O}$. Then, from Lemma 3.2, it follows that there exists a point $p : X \leftrightarrow \mathbb{1}$ such that $p \subseteq R;\overline{(S \cup S^{\mathsf{T}});\mathsf{L}} \subseteq R;\overline{S^{\mathsf{T}};\mathsf{L}}$ and $p \subseteq \overline{(S \cup S^{\mathsf{T}});\mathsf{L}} \subseteq \overline{S;\mathsf{L}}$. We now apply Theorem 3.1 to $p \subseteq R;\overline{S^{\mathsf{T}};\mathsf{L}}$ and get that there exists a point $q : X \leftrightarrow \mathbb{1}$ with $p;q^{\mathsf{T}} \subseteq R$ and $q \subseteq \overline{S^{\mathsf{T}};\mathsf{L}}$.

By (2) we have $S;\overline{S^{\mathsf{T}};\mathsf{L}} = \mathsf{O}$. This shows $p;q^{\mathsf{T}} \not\subseteq S$, since $p;q^{\mathsf{T}} \subseteq S$ would lead to the contradiction

$$p \subseteq S;q \subseteq S;\overline{S^{\mathsf{T}};\mathsf{L}} = \mathsf{O}\ .$$

As $p;q^{\mathsf{T}} \not\subseteq S$ is equivalent to $S \neq S \cup p;q^{\mathsf{T}}$, we have $S \subset S \cup p;q^{\mathsf{T}}$. To complete the proof, we now show that $S \cup p;q^{\mathsf{T}}$ is an $R$-matching. From $S \subseteq R$ and $p;q^{\mathsf{T}} \subseteq R$ we get $S \cup p;q^{\mathsf{T}} \subseteq R$. Next, the univalence of $S \cup p;q^{\mathsf{T}}$ is shown by the subsequent calculation:

$$\begin{aligned}
(S \cup p;q^{\mathsf{T}})^{\mathsf{T}};(S \cup p;q^{\mathsf{T}}) &= S^{\mathsf{T}};S \cup S^{\mathsf{T}};p;q^{\mathsf{T}} \cup q;p^{\mathsf{T}};S \cup q;p^{\mathsf{T}};p;q^{\mathsf{T}} &&\\
&\subseteq \mathsf{I} \cup S^{\mathsf{T}};p;q^{\mathsf{T}} \cup (S^{\mathsf{T}};p;q^{\mathsf{T}})^{\mathsf{T}} \cup q;p^{\mathsf{T}};p;q^{\mathsf{T}} && S \text{ is } R\text{-matching}\\
&= \mathsf{I} \cup S^{\mathsf{T}};p;q^{\mathsf{T}} \cup (S^{\mathsf{T}};p;q^{\mathsf{T}})^{\mathsf{T}} && \text{as } q;p^{\mathsf{T}};p;q^{\mathsf{T}} = q;\mathsf{L};q^{\mathsf{T}} = q;q^{\mathsf{T}} \subseteq \mathsf{I}\\
&= \mathsf{I} && \text{as } S^{\mathsf{T}};p;q^{\mathsf{T}} \subseteq S^{\mathsf{T}};\overline{S;\mathsf{L}};q^{\mathsf{T}} = \mathsf{O}\ .
\end{aligned}$$

Similarly, the injectivity of $S \cup p;q^{\mathsf{T}}$ can be shown in a using $S;q;p^{\mathsf{T}} \subseteq S;\overline{S^{\mathsf{T}};\mathsf{L}};p^{\mathsf{T}} = \mathsf{O}$ in the last step. Thus $S$ is not a maximal $R$-matching.

To prove the remaining property, we take an arbitrary maximal $R$-matching $S : X \leftrightarrow X$ and get then the desired result as follows:

$$\tau_R \leq \left|(S \cup S^{\mathsf{T}});\mathsf{L}_{X\mathbb{1}}\right| \leq |S;\mathsf{L}_{X\mathbb{1}}| + \left|S^{\mathsf{T}};\mathsf{L}_{X\mathbb{1}}\right| = |dom(S)| + |ran(S)| \leq 2{\cdot}|S| \leq 2{\cdot}\nu_R\ .$$

This calculation uses the first claim, axiom (C3) and results of Section 5 concerning domain and range, viz. estimate (6) and its analogon for the range. □

The estimates of the last two theorems have well-known graph-theoretic analogons. If $R : X \leftrightarrow X$ is the symmetric and irreflexive adjacency relation of an undirected graph $g = (X, E)$ and we call a subset $M$ of the edge set $E$ a *(graph-theoretic) matching* of $g$ if $e \cap f = \emptyset$, for all $e, f \in M$, then the two functions $M \mapsto \{ (x, y) \mid \{ x, y \} \in M \}$ and $S \mapsto \{ \{ x, y \} \mid (x, y) \in S \}$ constitute a one-to-one correspondence between the set of matchings of $g$ and the set of symmetric $R$-matchings. Since the step from the graph-theoretic matching to the associated symmetric $R$-matching doubles the cardinality, we get in the case of Theorem 7.8 that $2 \cdot \nu_g = \nu_R \leq 2 \cdot \tau_R$, where $\nu_g$ denotes the cardinality of a maximum matching of $g$, hence $\nu_g \leq \tau_R$. In case of Theorem 7.9 we use that the associated relational matching is symmetric. Then a maximal symmetric $R$-matching $S : X \leftrightarrow X$ leads to the $R$-vertex-cover $S;\mathsf{L} : X \leftrightarrow \mathbb{1}$ and the last calculation of the proof becomes $\tau_R \leq |S;\mathsf{L}_{X\mathbb{1}}| = |dom(S)| = |S| \leq \nu_R = 2 \cdot \nu_g$.

As the last graph parameter we consider the clique number. In an undirected graph $g = (X, E)$ a clique $C$ is a subset of the set $X$ of vertices such that for all $x, y \in C$ with $x \neq y$ it follows $\{ x, y \} \in E$. Translating this specification into the language of relation algebra, we define for a given symmetric and irreflexive relation $R : X \leftrightarrow X$ a vector $c : X \leftrightarrow \mathbb{1}$ to be an *$R$-clique* iff $c;c^{\mathsf{T}} \cap \overline{\mathsf{I}} \subseteq R$. Furthermore, we define the *clique number* of $R$ as

$$\omega_R := \max \{ |c| \mid c \text{ is an } R\text{-clique} \} .$$

In the last theorem of this section we show an estimate concerning the clique number $\omega_R$ and the chromatic number $\chi_R$.

**Theorem 7.10.** *For all symmetric and irreflexive relations $R : X \leftrightarrow X$, $R$-colourings $F : X \leftrightarrow X$ and $R$-cliques $c : X \leftrightarrow \mathbb{1}$ we have $|c| \leq |ran(F)|$ and $\omega_R \leq \chi_R$.*

*Proof.* First, we prove that all relations of the union $\bigcup_{p \in \mathcal{P}(c)} F^{\mathsf{T}};p$ are pairwise disjoint. To this end, assume $p, q : X \leftrightarrow \mathbb{1}$ to be arbitrary points such that $p \subseteq c$, $q \subseteq c$ and $p \neq q$. Then we have

$$\begin{aligned}
F^{\mathsf{T}};p &= F^{\mathsf{T}};p;q^{\mathsf{T}};q && \text{point property } q^{\mathsf{T}};q = \mathsf{L}_{\mathbb{1}\mathbb{1}} = \mathsf{I}_{\mathbb{1}} \\
&= F^{\mathsf{T}};\big(p;q^{\mathsf{T}} \cap \overline{\mathsf{I}}\big);q && \text{point property } p \neq q \iff p;q^{\mathsf{T}} \subseteq \overline{\mathsf{I}} \\
&\subseteq F^{\mathsf{T}};\big(c;c^{\mathsf{T}} \cap \overline{\mathsf{I}}\big);q && \text{since } p \subseteq c,\ q \subseteq c \\
&\subseteq F^{\mathsf{T}};R;q && c \text{ is an } R\text{-clique} \\
&= (R;F)^{\mathsf{T}};q && R \text{ is symmetric} \\
&\subseteq \overline{F}^{\mathsf{T}};q && \text{colouring property, transposition is monotonic} \\
&= \overline{F^{\mathsf{T}}};q && \text{property of the transposition} \\
&\subseteq \overline{F^{\mathsf{T}};q} && q \text{ is injective .}
\end{aligned}$$

Thus we have $F^{\mathsf{T}};p \cap F^{\mathsf{T}};q = \mathsf{O}$. Based on this preparatory result, the following calculation, yields first claim:

$$\begin{aligned}
|c| &= \textstyle\sum_{p \in \mathcal{P}(c)} 1 && \text{by Lemma 5.2} \\
&= \textstyle\sum_{p \in \mathcal{P}(c)} \big|F^{\mathsf{T}};p\big| && F \text{ mapping, Theorems 6.2 and 6.3} \\
&= \big|\textstyle\bigcup_{p \in \mathcal{P}(c)} F^{\mathsf{T}};p\big| && \text{by axioms } (C1), (C3), \text{preparatory result} \\
&= \big|F^{\mathsf{T}};\textstyle\bigcup_{p \in \mathcal{P}(c)} p\big| && \\
&= \big|F^{\mathsf{T}};c\big| && \text{by Lemma 3.1.(i)} \\
&\leq \big|F^{\mathsf{T}};\mathsf{L}\big| && \text{monotonicity of cardinality operation} \\
&= |ran(F)| .
\end{aligned}$$

Again, the second claim immediately follows from the first one by taking an $R$-colouring $F : X \leftrightarrow X$ with $|ran(F)| = \chi_R$ (with a minimum number of colours) and an $R$-clique $c : X \leftrightarrow \mathbb{1}$ with $|c| = \omega_R$ (a maximum one). □

In words, the equation $|c| = |F^{\mathsf{T}};c|$ of this proof means that the number of colours for colouring a clique is equal to the cardinality of the clique, and this is the main argument in the "traditional" informal proof that the clique number of a graph is less or equal its chromatic number.

## 8. A Calculational, Algebraic Proof of a Theorem of D. Kőnig

It is a well-known theorem of graph theory, published by D. Kőnig in [20], that for bipartite (undirected) graphs the matching number and the vertex cover number coincide. Relation-algebraically a bipartite (undirected) graph $g = (X, E)$ can be modelled by an (irreflexive) symmetric and bipartite relation, namely the adjacency relation of the graph. Thereby, a relation $R : X \leftrightarrow X$ is called *bipartite* if there exists a vector $v : X \leftrightarrow \mathbb{1}$ such that

$$(R;v \cap v) \cup (R;\overline{v} \cap \overline{v}) = \mathsf{O} \; . \tag{9}$$

The set $\{ v, \overline{v} \}$ of vectors is a partition of the universal vector $\mathsf{L}_{X\mathbb{1}} : X \leftrightarrow \mathbb{1}$. Since $\mathsf{L}_{X\mathbb{1}}$ models the entire set $X$, the subset $V$ of $X$ that is modelled by $v : X \leftrightarrow \mathbb{1}$, and the complement of the set $X \setminus V$ constitute a partition of $X$ in the traditional sense. In graph-theoretic terminology the above equation (9) specifies that $V$ and $X \setminus V$ are stable sets, that is, all edges of the graph $g$ connect only vertices from two different sets of the particular partition of $X$. The sets modelled by $v$ and $\overline{v}$ thus constitute a bipartition of the graph $g$ in the classical sense.

In Theorem 7.8 we proved that $\nu_R \leq 2\cdot\tau_R$, for all relations $R : X \leftrightarrow X$, thus also for symmetric and irreflexive ones. The proof of the theorem reveals that this estimate remains valid if $\nu_R$ denotes the cardinality of a maximum symmetric $R$-matching in case of a symmetric and irreflexive relation $R$. A modified specification of the *matching number for symmetric and irreflexive relations R* is given by

$$\nu^*_R := \max\{\, |S| \mid S = S^\mathsf{T} \text{ and } S \text{ is an } R\text{-matching} \,\} \; .$$

As discussed above we have $\nu^*_R \leq \nu_R \leq 2\cdot\tau_R$. In this section we show that for a bipartite, symmetric relation $R$ we have $\nu^*_R = 2\cdot\tau_R$. This result constitutes a relation-algebraic version of D. Kőnig's theorem; cf. the connection of graph-theoretic and symmetric relational matchings described in Section 7.

**Convention 8.1.** *For the remainder of this section we assume $R : X \leftrightarrow X$ to be a symmetric relation and $v : X \leftrightarrow \mathbb{1}$ to be a vector such that* (9) *holds, that is, $v$ specifies a bipartition of $R$. Furthermore, we assume $S : X \leftrightarrow X$ to be a symmetric $R$-matching.*

Note, that the prerequisite on $R$ and $v$ yields that both $R;v \cap v = \mathsf{O}$ and $R;\overline{v} \cap \overline{v} = \mathsf{O}$ hold, which by Boolean lattice rules yield the two inclusions $R;v \subseteq \overline{v}$ and $R;\overline{v} \subseteq v$, saying that $v$ and $\overline{v}$ are $R$-stable vectors. Based on the relations $R$ and $S$ and the vector $v$, we now introduce some auxiliary relations for our proof.

**Definition 8.1.** *Given the relations $R : X \leftrightarrow X$ and $S : X \leftrightarrow X$ and the vector $v : X \leftrightarrow \mathbb{1}$ of Convention 8.1, we define a subrelation $T : X \leftrightarrow X$ of $R$ and three vectors $u : X \leftrightarrow \mathbb{1}$, $r : X \leftrightarrow \mathbb{1}$ and $c : X \leftrightarrow \mathbb{1}$ as follows:*

$$T := R \cap \overline{S} \qquad u := \overline{S;\mathsf{L}} \qquad r := (\mathsf{I} \cup T);(S;T)^*;(u \cap v) \qquad c := (v \cap \overline{r}) \cup (\overline{v} \cap r) \; .$$

The relation $T$ is the relative complement of $S$ in $R$. The vector $u$ models the set of those vertices that are not contained in any matching edge (since $S$ is symmetric), which are called "uncovered" or "unmatched" in graph theory. Using again graph-theoretic terminology and the subset $V$ of $X$ to be modelled by $v$, the vertex set that is modelled by the vector $r$ is the set of all those vertices that are reachable from an uncovered vertex of $V$ along an $T$-$S$-alternating path, which is a path that traverses edges from $T$ and $S$ in alternating sequence. This vector is of utmost importance in the construction of a vertex cover, which is modelled by $c$. The above definitions are essentially the formal relation-algebraic descriptions of the informal specifications given in the textbook [12] when proving D. Kőnig's theorem. This entire construction originates from the work by D. Kőnig in [20]. However, both proofs are done in the usual "informal" mathematical manner, especially in a non-calculational and non-algebraic one.

Assuming $R$, $S$ and $v$ as introduced in Convention 8.1 and using $T$, $u$, $r$ and $c$ as introduced in Definition 8.1, in the remainder of this section we prove with calculational and algebraic means (including the cardinality axioms) that the vector $c$ is an $R$-vertex-cover and, if $S$ is a *maximum* symmetric $R$-matching, i.e., if additionally to the demands of Convention 8.1 the equation $|S| = \nu^*_R$ is satisfied, then $|S| = 2\cdot|c|$ holds. Note that the second statement yields the estimate $2\cdot\tau_R \leq 2\cdot|c| = |S| = \nu^*_R$, which proves, together with $\nu^*_R \leq 2\cdot\tau_R$, the desired result $\nu^*_R = 2\cdot\tau_R$.

At this point it should be mentioned that the equality $\nu^*_R = 2\cdot\tau_R$ can be obtained from a combination of some results presented in [16], but without the construction of an actual vertex cover as in the original version of Kőnig's theorem.

In [16], first Hall's theorem is shown (Theorem 3). Hall's theorem is used to prove a statement (Theorem 4), denoted as Kőnig's theorem and saying that for all relations $R : X \leftrightarrow Y$ with $\delta(R) > 0$ there exists a maximum relational matching $S : X \leftrightarrow Y$ (in the original sense of Section 6) such that $S \subseteq R$ and $|S| = |X| \cdot |Y| - \delta(R)$. Here $\delta(R)$ is defined by $\delta(R) := \max\{\, |R^{\mathsf{T}};p| - |p| \mid p : X \leftrightarrow \mathbb{1} \text{ point} \,\}$. That means that in Kawahara's version of Kőnig's theorem the graph-theoretic notion of a bipartition is not explicitly used, but modelled by the type of $R$, and nothing is said about the vertex cover number. Furthermore, it should be remarked that the equality of the matching number and the vertex cover number in case of bipartite relations is proved in a constructive fashion in [27], but, if cardinalities come into play, again with rather informal and non-algebraic means. Additionally, in [27] a different and more complicated definition of the concrete vertex cover is used.

We begin our proof with the following auxiliary statements. Each of the relation-algebraic formulae describes a graph-theoretic fact that is used in the informal graph-theoretic proof of D. Kőnig's theorem in [12].

**Lemma 8.1.** *For the relations R, S, T and the vectors r, u, v from Convention 8.1 and Definition 8.1 we have:*

*(i)* $S;(S;T)^*;(u \cap v) \subseteq T;(S;T)^*;(u \cap v)$.

*(ii)* $R;(S;T)^*;(u \cap v) = T;(S;T)^*;(u \cap v)$.

*(iii)* $v \cap r = (S;T)^*;(u \cap v)$.

*(iv)* $\overline{v} \cap r = T;(S;T)^*;(u \cap v)$.

*Proof.* (i) Using (2) and the symmetry of $S$ we obtain $\mathsf{O} = S^{\mathsf{T}};\overline{S;\mathsf{L}} = S;\overline{S;\mathsf{L}}$. Thus we get:

$$\begin{aligned} S;(S;T)^*;(u \cap v) &= S;(\mathsf{I} \cup S;T;(S;T)^*);(u \cap v) && \text{closure property} \\ &= (S \cup S;S;T;(S;T)^*);(u \cap v) \\ &\subseteq (S \cup T;(S;T)^*);(u \cap v) && S;S = S;S^{\mathsf{T}} \subseteq \mathsf{I} \\ &= S;(u \cap v) \cup T;(S;T)^*;(u \cap v) \\ &= T;(S;T)^*;(u \cap v) && \text{as } S;(u \cap v) \subseteq S;u = S;\overline{S;\mathsf{L}} = \mathsf{O}\,. \end{aligned}$$

(ii) We calculate as follows to show the desired result:

$$\begin{aligned} R;(S;T)^*;(u \cap v) &= (T \cup S);(S;T)^*;(u \cap v) && T = R \cap \overline{S} \text{ by definition of } T \text{ and } S \subseteq R \\ &= T;(S;T)^*;(u \cap v) \cup S;(S;T)^*;(u \cap v) \\ &\subseteq T;(S;T)^*;(u \cap v) \cup T;(S;T)^*;(u \cap v) && \text{by Lemma 8.1.(i)} \\ &= T;(S;T)^*;(u \cap v) \\ &\subseteq R;(S;T)^*;(u \cap v) && \text{since } T \subseteq R\,. \end{aligned}$$

(iii) The $R$-stability of $v$ and $\overline{v}$ yields $R;R;v \subseteq R;\overline{v} \subseteq v$. By induction, we get $(R;R)^k;v \subseteq v$, for all $k \in \mathbb{N}$, which yields $(R;R)^*;v \subseteq v$. In particular, we obtain

$$T;(S;T)^*;(u \cap v) \subseteq T;(S;T)^*;v \subseteq R;(R;R)^*;v \subseteq R;v \subseteq \overline{v} \tag{10}$$

and thus $T;(S;T)^*;(u \cap v) \cap v = \mathsf{O}$. Also, we get:

$$(S;T)^*;(u \cap v) \subseteq (R;R)^*;v \subseteq v \tag{11}$$

This allows to prove the claim as follows:

$$\begin{aligned} v \cap r &= v \cap (\mathsf{I} \cup T);(S;T)^*;(u \cap v) && \text{definition of } r \\ &= v \cap ((S;T)^*;(u \cap v) \cup T;(S;T)^*;(u \cap v)) \\ &= (v \cap (S;T)^*;(u \cap v)) \cup (v \cap T;(S;T)^*;(u \cap v)) \\ &= v \cap (S;T)^*;(u \cap v) && \text{second intersection is } \mathsf{O} \\ &= (S;T)^*;(u \cap v) && \text{by (11)}\,. \end{aligned}$$

(iv) Using the inclusion (11) we get $(S;T)^*;(u \cap v) \cap \overline{v} = \mathsf{O}$. With this equality we now conclude the proof by the subsequent calculation:

$$\begin{aligned}
\overline{v} \cap r &= \overline{v} \cap (\mathsf{I} \cup T);(S;T)^*;(u \cap v) && \text{definition of } r \\
&= \overline{v} \cap ((S;T)^*;(u \cap v) \cup T;(S;T)^*;(u \cap v)) \\
&= (\overline{v} \cap (S;T)^*;(u \cap v)) \cup (\overline{v} \cap T;(S;T)^*;(u \cap v)) \\
&= \overline{v} \cap T;(S;T)^*;(u \cap v) && \text{first intersection is } \mathsf{O} \\
&= T;(S;T)^*;(u \cap v) && \text{by (10)} .
\end{aligned}$$

□

Using the definition of the vector $c$ in the first step, we have

$$\overline{c} = \overline{(v \cap \overline{r}) \cup (\overline{v} \cap r)} = (\overline{v} \cup r) \cap (v \cup \overline{r}) = (\overline{v} \cap v) \cup (\overline{v} \cap \overline{r}) \cup (r \cap v) \cup (r \cap \overline{r}) = (v \cap r) \cup (\overline{v} \cap \overline{r}) . \tag{12}$$

With these prerequisites we can now show that the vector $c$ is in fact an $R$-vertex-cover. The first two results of the following lemma are auxiliary statements, which are needed in the proof of the third result, the statement we are actually interested in.

**Lemma 8.2.** *For the relation $R$ and the vectors $c$, $r$, $v$ from Convention 8.1 and Definition 8.1 we have:*

*(i)* $R;(v \cap r) = \overline{v} \cap r$.

*(ii)* $R;(\overline{v} \cap \overline{r}) \subseteq v \cap \overline{r}$.

*(iii)* *$c$ is an $R$-vertex-cover, i.e.,* $R;\overline{c} \subseteq c$.

*Proof.* (i) This statement can be shown as follows:

$$\begin{aligned}
R;(v \cap r) &= R;(S;T)^*;(u \cap v) && \text{by Lemma 8.1.(iii)} \\
&= T;(S;T)^*;(u \cap v) && \text{by Lemma 8.1.(ii)} \\
&= \overline{v} \cap r . && \text{by Lemma 8.1.(iv)}
\end{aligned}$$

(ii) We first observe that the following equivalences hold due to a Schröder rule and $R = R^\mathsf{T}$:

$$R;(\overline{v} \cap \overline{r}) \subseteq v \cap \overline{r} \iff R^\mathsf{T};\overline{v \cap \overline{r}} \subseteq \overline{\overline{v} \cap \overline{r}} \iff R;(\overline{v} \cup r) \subseteq v \cup r .$$

Now we show that $R;(\overline{v} \cup r) \subseteq v \cup r$ holds, which completes the proof. To reach the goal, we calculate as follows:

$$\begin{aligned}
R;(\overline{v} \cup r) &= R;\overline{v} \cup R;r \\
&\subseteq v \cup R;r && \text{as } \overline{v} \text{ is } R\text{-stable} \\
&= v \cup R;((v \cap r) \cup (\overline{v} \cap r)) && \text{as } (v \cap r) \cup (\overline{v} \cap r) = r \\
&= v \cup R;(v \cap r) \cup R;(\overline{v} \cap r) \\
&\subseteq v \cup (\overline{v} \cap r) \cup R;(\overline{v} \cap r) && \text{by Lemma 8.2.(i)} \\
&\subseteq v \cup (\overline{v} \cap r) \cup v && R;(\overline{v} \cap r) \subseteq R;\overline{v} \subseteq v \text{ as } \overline{v} \text{ is } R\text{-stable} \\
&= v \cup (\overline{v} \cap r) \\
&= (v \cup \overline{v}) \cap (v \cup r) \\
&= v \cup r .
\end{aligned}$$

(iii) Because of the first two results, this is now simple. We get

$$R;\overline{c} = R;((v \cap r) \cup (\overline{v} \cap \overline{r})) = R;(v \cap r) \cup R;(\overline{v} \cap \overline{r}) \subseteq (\overline{v} \cap r) \cup (v \cap \overline{r}) = c$$

using property (12), then Lemma 8.2.(i) and Lemma 8.2.(ii) and, finally, the definition of $c$. □

Having shown that the vector $c$ is an $R$-vertex cover, we now prove two facts concerning the relation $S$ and the vectors $r$ and $c$. Again the first result of the following lemma constitutes an auxiliary result for the second one, the statement we are actually interested in, viz. the $S$-stability of the $R$-vertex-cover $c$.

**Lemma 8.3.** *For the relation $S$ and the vectors $r, c$ from Convention 8.1 and Definition 8.1 we have:*

*(i)* $S;r \subseteq r$.

*(ii)* $S;c \cap c = \mathsf{O}$.

*Proof.* (i) To show the claim we calculate as follows:

$$\begin{aligned} S;r &= S;(\mathsf{I} \cup T);(S;T)^*;(u \cap v) && \text{definition of } r \\ &= S;(S;T)^*;(u \cap v) \cup S;T;(S;T)^*;(u \cap v) \\ &\subseteq T;(S;T)^*;(u \cap v) \cup S;T;(S;T)^*;(u \cap v) && \text{by Lemma 8.1.(i)} \\ &\subseteq T;(S;T)^*;(u \cap v) \cup (S;T)^*;(u \cap v) && \text{closure property} \\ &= (T \cup \mathsf{I});(S;T)^*;(u \cap v) \\ &= r && \text{definition of } r\,. \end{aligned}$$

(ii) We show the $S$-stability $S;c \subseteq \overline{c}$ of $c$, which is equivalent to the desired equality by Boolean lattice rules. The inclusion is shown by the following calculation:

$$\begin{aligned} S;c &= S;((v \cap \overline{r}) \cup (\overline{v} \cap r))) && \text{definition of } c \\ &= S;(v \cap \overline{r}) \cup S;(\overline{v} \cap r) \\ &\subseteq (S;v \cap S;\overline{r}) \cup (S;\overline{v} \cap S;r) \\ &\subseteq (\overline{v} \cap S;\overline{r}) \cup (v \cap S;r) && \text{as } v \text{ and } \overline{v} \text{ are } R\text{-stable, } S \subseteq R \\ &\subseteq (\overline{v} \cap \overline{r}) \cup (v \cap r) && \text{see below} \\ &= \overline{c} && \text{by (12)}\,. \end{aligned}$$

Because of Lemma 8.3.(i) we have $S;r \subseteq r$, which by a Schröder rule and $S = S^{\mathsf{T}}$ is equivalent to $S;\overline{r} \subseteq \overline{r}$. □

For the remainder of this section we assume that $S$ is a maximum symmetric $R$-matching. This pre-condition is important for the cardinality calculation: while a non-maximum matching yields a vertex cover, this vertex cover may not have minimum cardinality. The key idea in the calculation of the cardinality of $c$ is to show that $\overline{u}$ is the disjoint union of the vectors $c$ and $S;c$. One important step in this calculation is the following property of maximum matchings.

**Theorem 8.1.** *If the relation $S$ from Convention 8.1 is a maximum symmetric $R$-matching, then for the relation $T$ and the vector $u$ from Definition 8.1 we have $T;(S;T)^*;u \subseteq \overline{u}$.*

In [9] this theorem is shown with non-algebraic means. A relation-algebraic proof that follows this idea is tedious. Therefore, we have transferred it into an appendix. Theorem 8.1 can be seen as relation-algebraic version of one direction of C. Berge's well-known characterisation of maximum graph-theoretic matchings given in [1]. Note, however, that the inclusion $T;(S;T)^*;u \subseteq \overline{u}$ of the theorem means that there exists no $T$-$S$-alternating path between uncovered vertices, where in such paths multiple occurrences of vertices are possible, whereas in [1] multiple occurrences of vertices in alternating paths are forbidden. After these preparations we next prove the desired decomposition of $\overline{u}$. As in the case of Lemma 8.1 each of the relation-algebraic formulae again describes a graph-theoretic fact that is used in the proof of [12].

**Lemma 8.4.** *If the relation $S$ from Convention 8.1 is a maximum symmetric $R$-matching, then for the vectors $c$, $u$ from Definition 8.1 we have:*

*(i)* $c \subseteq \overline{u} = S;\mathsf{L}$.

*(ii)* $S;c \cup c = S;\mathsf{L}$.

*Proof.* (i) Using that reflexive-transitive closures are reflexive and the definition of the vector $r$ we get

$$u \cap v = \mathsf{I};(u \cap v) \subseteq \mathsf{I};(S;T)^*;(u \cap v) \subseteq (\mathsf{I} \cup T);(S;T)^*;(u \cap v) = r$$

for the vectors $r$, $u$ and $v$ which, in turn, is equivalent to the inclusion $v \cap \overline{r} \subseteq \overline{u}$ by a well-known Boolean lattice rule. With this auxiliary result we obtain the desired inclusion as follows:

$$\begin{array}{rcll} c & = & (v \cap \overline{r}) \cup (\overline{v} \cap r) & \text{definition of } c \\ & \subseteq & \overline{u} \cup (\overline{v} \cap r) & \text{as } v \cap \overline{r} \subseteq \overline{u} \\ & = & \overline{u} \cup T;(S;T)^*;(u \cap v) & \text{by Lemma 8.1.(iv)} \\ & \subseteq & \overline{u} \cup T;(S;T)^*;u & \\ & \subseteq & \overline{u} \cup \overline{u} & \text{by Theorem 8.1 } (S \text{ maximum matching}) \\ & = & S;\mathsf{L} & \text{definition of } u\,. \end{array}$$

(ii) Here we calculate as follows to prove the claim:

$$\begin{array}{rcll} S;\mathsf{L} & = & S;(c \cup \overline{c}) & \\ & = & S;c \cup S;\overline{c} & \\ & \subseteq & S;c \cup R;\overline{c} & \text{since } S \subseteq R \\ & \subseteq & S;c \cup c & c \text{ is an } R\text{-vertex-cover by Lemma 8.2.(iii)} \\ & \subseteq & S;c \cup S;\mathsf{L} & \text{by Lemma 8.4.(i)} \\ & = & S;\mathsf{L} & \text{since } c \subseteq \mathsf{L}\,. \end{array}$$ □

We are now in the position to prove that $2{\cdot}|c| = |S|$ holds for the symmetric $R$-matching $S$ and the vector $c$ if $S$ is a maximum symmetric $R$-matching, from which, as shown at the beginning of this section, the desired result $\nu^*_R = 2{\cdot}\tau_R$ follows. This is the place where two cardinality axioms and two former results of Section 6 on the cardinality of mapping-related relations come into play.

**Theorem 8.2.** *If the relation $S$ of Convention 8.1 is a maximum symmetric $R$-matching, then we have $2 \cdot |c| = |S|$, where $c$ is the vector from Definition 8.1.*

*Proof.* The following calculation shows the claim:

$$\begin{array}{rcll} |S| & = & |S;\mathsf{L}_{X\mathbb{1}}| & \text{by Lemma 4.1.(ii) } (S \text{ univalent, } \mathsf{L}_{X\mathbb{1}} \text{ mapping}) \\ & = & |S;c \cup c| & \text{by Lemma 8.4.(ii)} \\ & = & |S;c| + |c| - |S;c \cap c| & \text{by axiom (C3)} \\ & = & |S;c| + |c| - |\mathsf{O}| & \text{by Lemma 8.3.(ii)} \\ & = & |S;c| + |c| & \text{by axiom (C1)} \\ & = & \left|S^{\mathsf{T}};c\right| + |c| & S \text{ is symmetric} \\ & = & |c| + |c| & \text{by Theorem 4.2.(iii) } (S \text{ univalent, injective, } c \subseteq S;\mathsf{L} \text{ by Lemma 8.4.(i)})\,. \end{array}$$ □

## 9. Concluding Remarks

As we have mentioned in the introduction, the present paper is a continuation of [16] and [5]. We have extended the stock of fundamental properties of cardinalities of relations by results that concern vectors, points and mapping-related relations. As applications of these results we have verified properties of linear orders and graphs in a very precise and calculational manner. The latter include the cardinalities of rooted trees, well-known estimates of graph parameters and a proof of the fact that in bipartite graphs the matching number equals the vertex cover number.

Concerning the last result, the reader may wonder whether there is any merit of having a somewhat long and technical relation-algebraic proof of a well-known graph-theoretic statement. Especially the proof of the statement that the existence of an augmenting path yields a larger matching is usually much shorter in literature. Note, however, that this is due to a large degree of informality. In the above example, one usually simply states that a certain set is a larger matching, without a (formal) mathematical proof. Providing all missing arguments in such cases in detail would also lead to longer and more precise proofs. Additionally, the restriction to a fixed set of axioms and a calculational approach provide less room for errors and open, as we have mentioned in the introduction, the possibility for proof mechanisation that enhances reliability once again. In cooperation with D. Pous we have extended the relation-algebra library of [21] for the proof assistant Coq by the singleton set axioms, the point axiom, Kawahara's cardinality axioms and the relational axiom of choice and some decision procedures concerning cardinalities and have already verified many of the results of Section 4 to 6 by means of Coq. Our goal is a complete verification of the results of the present paper via this tool.

For the future we also plan to extend the presented results about cardinalities to further important classes of relations and also to relation-algebraic constructions that frequently appear in practical applications. Examples for the first are certain types of orders used in decision theory and preference modeling, examples for the latter are residuals and symmetric quotients. We also plan applications in the development and verification of such relational programs where cardinalities play a fundamental role. Following the lines of [5] this includes further approximation algorithms. We also think of randomised algorithms, since such algorithms typically use random bits as an auxiliary input and these can be seen as a random relational vector of a certain cardinality. Experience has shown that RelView supports the development of relational algorithms in a significant manner. But in the programming language of the present Version 8.2 of the tool there are only operations available which compare cardinalities of relations, e.g., $cardeq(R, S)$ for testing $|R| = |S|$. There exists no cardinality operation in the sense of the present paper. To extend the usability of the tool we, therefore, plan the inclusion of such an operation.

**Acknowledgement.** We thank W. Guttmann and M. Winter for their valuable remarks and D. Pous for his cooperation concerning proof mechanisation. NICTA is funded by the Australian Government through the Department of Communications and the Australian Research Council through the ICT Centre of Excellence Program.

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## Appendix A. Proof of Theorem 8.1

Let us rephrase the statement we wish to prove. Assume $R : X \leftrightarrow X$ to be symmetric, $S \subseteq R$ to be a maximum symmetric $R$-matching and $T$ and $u$ to be defined as $T := R \cap \overline{S}$ and $u := \overline{S;\mathsf{L}_{X\mathbb{1}}}$. Then we have $T;(S;T)^*;u \subseteq \overline{u}$.

The following relation-algebraic proof is based upon the original proof of [1]. In particular, we use essentially the same construction and prove the same auxiliary statements. We use contraposition. So, assume $T;(S;T)^*;u \cap u \neq \mathsf{O}$. Then by Lemma 3.2 this vector contains a point $p : X \leftrightarrow \mathbb{1}$. Since $p \subseteq T;(S;T)^*;u$, by Theorem 3.1 there exists a point $q : X \leftrightarrow \mathbb{1}$ such that $p;q^\mathsf{T} \subseteq T;(S;T)^*$, i.e., $p \subseteq T;(S;T)^*;q$, and $q \subseteq u$. Using Lemma 3.2 again, we get a point $z : X \leftrightarrow \mathbb{1}$, such that $p \subseteq T;z$ and $z \subseteq (S;T)^*;q$, i.e., $p;z^\mathsf{T} \subseteq T$ and similarly $z;q^\mathsf{T} \subseteq (S;T)^*$. Since $z;q^\mathsf{T}$ is an atom among the relations of type $X \leftrightarrow X$ and $(S;T)^* = \bigcup_{i\in\mathbb{N}}(S;T)^i$, there exists a minimal $n \in \mathbb{N}$ such that $z;q^\mathsf{T} \subseteq (S;T)^n$. In [27] it is shown that this yields a sequence of points $(w_0, \ldots, w_n)$ such that $w_0 = z$, $w_n = q$ and $w_i \subseteq S;T;w_{i+1}$, for all $i \in \mathbb{N}_{<n}$. For all $i \in \mathbb{N}_{<n}$ the intermediate point theorem (which we have discussed after Lemma 3.2) yields a point $x_i : X \leftrightarrow \mathbb{1}$ such that $w_i \subseteq S;x_i$ and $x_i \subseteq T;w_{i+1}$, i.e., $w_i;x_i{}^\mathsf{T} \subseteq S$ and $x_i;w_{i+1}{}^\mathsf{T} \subseteq T$. We now define a sequence $\alpha := (\alpha_0, \ldots, \alpha_{2n+1})$ by

$$\alpha_i := \begin{cases} p & : i = 0 \\ w_{\frac{i-1}{2}} & : i \neq 0 \wedge i \text{ odd} \\ x_{\frac{i}{2}-1} & : i \neq 0 \wedge i \text{ even} \end{cases}$$

for all $i \in \mathbb{N}_{<2n+2}$, i.e., the sequence $(p, w_0, x_0, \ldots, w_{n-1}, x_{n-1}, w_n)$. The intuition behind this sequence is that we have unfolded a path in the graph with relation $S;T$.

**Lemma A.1.** *(i) For all $i \in \mathbb{N}_{<2n+1}$ we have $\alpha_i;\alpha_{i+1}{}^\mathsf{T} \subseteq T$ if $i$ is even and $\alpha_i;\alpha_{i+1}{}^\mathsf{T} \subseteq S$ if $i$ is odd.*

*(ii) For all $j, k \in \mathbb{N}_{<2n+2}$ we have that $\alpha_j = \alpha_k$ implies $j = k$.*

*Proof.* (i) Let $i \in \mathbb{N}_{<2n+1}$. If $i = 0$, we obtain the desired inclusion

$$\alpha_i;\alpha_{i+1}{}^\mathsf{T} = \alpha_0;\alpha_1{}^\mathsf{T} = p;w_0{}^\mathsf{T} = p;z^\mathsf{T} \subseteq T$$

by the choice of the intermediate points discussed above. If $i > 0$ and $i$ is odd, we get

$$\alpha_i;\alpha_{i+1}{}^\mathsf{T} = w_{\frac{i-1}{2}};x_{\frac{i+1}{2}-1}{}^\mathsf{T} = w_{\frac{i-1}{2}};x_{\frac{i-1}{2}}{}^\mathsf{T} \subseteq S\ ,$$

and if $i > 0$ and $i$ is even, we obtain the claim in a similar fashion.

(ii) Let $j, k \in \mathbb{N}_{<2n+2}$ such that $\alpha_j = \alpha_k$. W.l.o.g. we may assume $j \leq k$ and $\alpha_j \subseteq v$. From (i) we get $\alpha_j;\alpha_k{}^\mathsf{T} \subseteq R^{k-j}$ and thus $\alpha_j \subseteq R^{k-j};\alpha_k$. A simple induction shows $R^{2m};v \subseteq v$ and $R^{2m+1};v \subseteq \overline{v}$, for all $m \in \mathbb{N}$, as $\{v, \overline{v}\}$ is a bipartition of $R$. If $k - j$ was odd, we would obtain $\alpha_j \subseteq v \cap R^{k-j};\alpha_k \subseteq v \cap R^{k-j};v \subseteq v \cap \overline{v} = \mathsf{O}$, which contradicts the fact that $\alpha_j$ is a point. So, $k - j$ is even and thus either both $j, k$ are odd or both are even. If $j, k$ are odd, we get:

$$\begin{aligned} z;q^\mathsf{T} &= \alpha_1;\alpha_j{}^\mathsf{T};\alpha_j;\alpha_{2n+1}{}^\mathsf{T} && \text{as } \alpha_j{}^\mathsf{T};\alpha_j = \mathsf{L}_{\mathbb{1}\mathbb{1}} = \mathsf{I}_{\mathbb{1}} \text{ by point property and Axiom 2.2} \\ &= \left(\textstyle\prod_{l=1}^{j-1} \alpha_l;\alpha_{l+1}{}^\mathsf{T}\right);\left(\textstyle\prod_{l=k}^{2n} \alpha_l;\alpha_{l+1}{}^\mathsf{T}\right) && \text{recursive application of the previous argument} \\ &\subseteq (S;T)^{\frac{j-1}{2}};(S;T)^{\frac{2n-k+1}{2}} && \alpha_l;\alpha_{l+1}{}^\mathsf{T} \subseteq S \text{ if } l \text{ odd, } \alpha_l;\alpha_{l+1}{}^\mathsf{T} \subseteq T \text{ if } l \text{ even, } j-1 \text{ and } 2n \text{ even} \\ &= (S;T)^{n-\frac{k-j}{2}}\ . \end{aligned}$$

Since $n$ is minimal such that $z;q^{\mathsf{T}} \subseteq (S;T)^n$, we get $\frac{k-j}{2} = 0$, hence $k = j$. If $j, k$ are both even, then we obtain:

$$\begin{aligned} z;q^{\mathsf{T}} &= \left(\prod_{l=1}^{j-1} \alpha_l;\alpha_{l+1}{}^{\mathsf{T}}\right);\left(\prod_{l=k}^{2n} \alpha_l;\alpha_{l+1}{}^{\mathsf{T}}\right) && \text{see first case} \\ &\subseteq (S;T)^{\frac{j}{2}-1};S;T;(S;T)^{\frac{2n-k}{2}} && \alpha_l;\alpha_{l+1}{}^{\mathsf{T}} \subseteq S \text{ if } l \text{ odd, } \alpha_l;\alpha_{l+1}{}^{\mathsf{T}} \subseteq T \text{ if } l \text{ even, } j-1 \text{ odd, } k \text{ even} \\ &= (S;T)^{\frac{2n-k+j}{2}} . \end{aligned}$$

Due to $\frac{2n-k+j}{2} = n - \frac{k-j}{2}$, once again the minimality of $n$ yields that $k = j$. □

Using classical graph-theoretic terminology, the sequence $\alpha$ models an $S$-augmenting path, which is to say that it can be used to construct another matching with a strictly larger cardinality. We now prove this very statement using purely algebraic means. To this end, we define the following auxiliary relations:

$$C := \bigcup_{i\in\mathbb{N}_{<2n+1}} \alpha_i;\alpha_{i+1}{}^{\mathsf{T}} \qquad D := C \cup C^{\mathsf{T}} \qquad S_+ := (S \cap \overline{D}) \cup (D \cap \overline{S}) . \tag{A.1}$$

In the classical sense $D$ can be considered the set of all edges along the augmenting path modeled by $\alpha$ and the symmetric difference $S_+$ of $S$ and $D$ constitutes a symmetric $R$-matching that contains one edge more than $S$ (which in our terminology states that the new matching contains two more elements than $S$). To prove the latter fact in the remainder of this section, first, we show that the two relations whose union is $S_+$ have very convenient representations.

**Lemma A.2.**

*(i)* $S \cap \overline{D} = S \cap \overline{\left(\bigcup_{i\in\mathbb{N}_{<n}} \alpha_{2i+1};\alpha_{2i+2}{}^{\mathsf{T}}\right) \cup \left(\bigcup_{i\in\mathbb{N}_{<n}} \alpha_{2i+2};\alpha_{2i+1}{}^{\mathsf{T}}\right)} = S \cap \overline{\bigcup_{i\in\mathbb{N}_{<n}} \alpha_{2i+1};\alpha_{2i+2}{}^{\mathsf{T}} \cup \alpha_{2i+2};\alpha_{2i+1}{}^{\mathsf{T}}}$.

*(ii)* $D \cap \overline{S} = \left(\bigcup_{i\in\mathbb{N}_{<n+1}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}}\right) \cup \left(\bigcup_{i\in\mathbb{N}_{<n+1}} \alpha_{2i+1};\alpha_{2i}{}^{\mathsf{T}}\right) = \bigcup_{i\in\mathbb{N}_{<n+1}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}} \cup \alpha_{2i+1};\alpha_{2i}{}^{\mathsf{T}}$.

*Proof.* (i) For all $i \in \mathbb{N}_{<n}$ we have $\alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}} \subseteq T$ by Lemma A.1.(i). This yields $\bigcup_{i\in\mathbb{N}_{<n}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}} \subseteq \overline{S}$ and thus $S \subseteq \overline{\bigcup_{i\in\mathbb{N}_{<n}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}}}$. Now we calculate as follows:

$$\begin{aligned} S \cap \overline{C} &= S \cap \overline{\bigcup_{i\in\mathbb{N}_{<2n+1}} \alpha_i;\alpha_{i+1}{}^{\mathsf{T}}} && \text{definition of } C \text{ by (A.1)} \\ &= S \cap \overline{\left(\bigcup_{i\in\mathbb{N}_{<n+1}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}}\right) \cup \left(\bigcup_{i\in\mathbb{N}_{<n}} \alpha_{2i+1};\alpha_{2i+2}{}^{\mathsf{T}}\right)} && \text{splitting into even and odd indices} \\ &= S \cap \overline{\bigcup_{i\in\mathbb{N}_{<n+1}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}}} \cap \overline{\bigcup_{i\in\mathbb{N}_{<n}} \alpha_{2i+1};\alpha_{2i+2}{}^{\mathsf{T}}} \\ &= S \cap \overline{\bigcup_{i\in\mathbb{N}_{<n}} \alpha_{2i+1};\alpha_{2i+2}{}^{\mathsf{T}}} && \text{see above .} \end{aligned}$$

With the above calculation we get:

$$\begin{aligned} S \cap \overline{D} &= S \cap \overline{C \cup C^{\mathsf{T}}} && \text{definition of } D \text{ by (A.1)} \\ &= \left(S \cap \overline{C}\right) \cap \left(S \cap \overline{C}\right)^{\mathsf{T}} && S \text{ is symmetric} \\ &= \left(S \cap \overline{\bigcup_{i\in\mathbb{N}_{<n}} \alpha_{2i+1};\alpha_{2i+2}{}^{\mathsf{T}}}\right) \cap \left(S \cap \overline{\bigcup_{i\in\mathbb{N}_{<n}} \alpha_{2i+1};\alpha_{2i+2}{}^{\mathsf{T}}}\right)^{\mathsf{T}} && \text{see above} \\ &= S \cap \overline{\bigcup_{i\in\mathbb{N}_{<n}} \alpha_{2i+1};\alpha_{2i+2}{}^{\mathsf{T}}} \cap S \cap \overline{\bigcup_{i\in\mathbb{N}_{<n}} \alpha_{2i+2};\alpha_{2i+1}{}^{\mathsf{T}}} && S \text{ is symmetric} \\ &= S \cap \overline{\left(\bigcup_{i\in\mathbb{N}_{<n}} \alpha_{2i+1};\alpha_{2i+2}{}^{\mathsf{T}}\right) \cup \left(\bigcup_{i\in\mathbb{N}_{<n}} \alpha_{2i+2};\alpha_{2i+1}{}^{\mathsf{T}}\right)} \\ &= S \cap \overline{\bigcup_{i\in\mathbb{N}_{<n}} \alpha_{2i+1};\alpha_{2i+2}{}^{\mathsf{T}} \cup \alpha_{2i+2};\alpha_{2i+1}{}^{\mathsf{T}}} . \end{aligned}$$

(ii) For all $i \in \mathbb{N}_{<n+1}$ we have $\alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}} \subseteq T \subseteq \overline{S}$ and $\alpha_{2i+1};\alpha_{2i+2}{}^{\mathsf{T}} \subseteq S$ by Lemma A.1.(i). As a consequence we get the inclusions $\bigcup_{i\in\mathbb{N}_{<n+1}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}} \subseteq \overline{S}$ and $\bigcup_{i\in\mathbb{N}_{<n}} \alpha_{2i+1};\alpha_{2i+2}{}^{\mathsf{T}} \subseteq S$. Next, we calculate as follows:

$$\begin{aligned} C \cap \overline{S} &= \left(\bigcup_{i\in\mathbb{N}_{<2n+1}} \alpha_i;\alpha_{i+1}{}^{\mathsf{T}}\right) \cap \overline{S} && \text{definition of } C \text{ by (A.1)} \\ &= \left(\left(\bigcup_{i\in\mathbb{N}_{<n+1}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}}\right) \cup \left(\bigcup_{i\in\mathbb{N}_{<n}} \alpha_{2i+1};\alpha_{2i+2}{}^{\mathsf{T}}\right)\right) \cap \overline{S} && \text{splitting into even and odd indices} \end{aligned}$$

$$= \left(\left(\bigcup_{i\in\mathbb{N}_{<n+1}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}}\right) \cap \overline{S}\right) \cup \left(\left(\bigcup_{i\in\mathbb{N}_{<n}} \alpha_{2i+1};\alpha_{2i+2}{}^{\mathsf{T}}\right) \cap \overline{S}\right)$$

$$= \bigcup_{i\in\mathbb{N}_{<n+1}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}} \qquad \text{see above .}$$

Now we are able to conclude the proof as follows:

$$\begin{aligned}
D \cap \overline{S} &= \left(C \cup C^{\mathsf{T}}\right) \cap \overline{S} && \text{definition of } D \text{ by (A.1)}\\
&= \left(C \cap \overline{S}\right) \cup \left(C \cap \overline{S}\right)^{\mathsf{T}} && S \text{ is symmetric}\\
&= \left(\bigcup_{i\in\mathbb{N}_{<n+1}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}}\right) \cup \left(\bigcup_{i\in\mathbb{N}_{<n+1}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}}\right)^{\mathsf{T}} && \text{see above}\\
&= \left(\bigcup_{i\in\mathbb{N}_{<n+1}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}}\right) \cup \left(\bigcup_{i\in\mathbb{N}_{<n+1}} \alpha_{2i+1};\alpha_{2i}{}^{\mathsf{T}}\right)\\
&= \bigcup_{i\in\mathbb{N}_{<n+1}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}} \cup \alpha_{2i+1};\alpha_{2i}{}^{\mathsf{T}} .
\end{aligned}$$

□

To show that $S_+$ is a symmetric $R$-matching and larger than $S$, we we need two further auxiliary results.

**Lemma A.3.**

*(i) For all $i \in \mathbb{N}_{<2n+2}$ we have* $\left(S \cap \overline{D}\right);\alpha_i = \mathsf{O}$

*(ii) We have* $\left(S \cap \overline{D}\right);\left(D \cap \overline{S}\right) = \mathsf{O}$.

*Proof.* (i) We use contradiction and assume to the contrary that there exists $i \in \mathbb{N}_{<2n+2}$ with $S;\alpha_i \cap \overline{D};\alpha_i \neq \mathsf{O}$. Then Lemma 3.2 yields a point $x : X \leftrightarrow \mathbb{1}$ such that $x \subseteq S;\alpha_i$ and $x \subseteq \overline{D};\alpha_i$. Thus, we get $x;\alpha_i{}^{\mathsf{T}} \subseteq S$ and $x;\alpha_i{}^{\mathsf{T}} \subseteq \overline{D}$. If $i$ is odd, we have $\alpha_i;\alpha_{i+1}{}^{\mathsf{T}} \subseteq S$ by Lemma A.1.(i). As a consequence, $x;\alpha_{i+1}{}^{\mathsf{T}} = x;\alpha_i{}^{\mathsf{T}};\alpha_i;\alpha_{i+1}{}^{\mathsf{T}} \subseteq S;S \subseteq \mathsf{I}$. The point property yields that $x \subseteq \mathsf{I};\alpha_{i+1} = \alpha_{i+1}$ and thus $x = \alpha_{i+1}$. Now we have $\alpha_i;\alpha_{i+1}{}^{\mathsf{T}} \subseteq D$ by (A.1), but at the same time

$$\alpha_i;\alpha_{i+1}{}^{\mathsf{T}} = \alpha_i;x^{\mathsf{T}} = (x;\alpha_i{}^{\mathsf{T}})^{\mathsf{T}} \subseteq \overline{D}^{\mathsf{T}} = \overline{D} .$$

This is a contradiction, because $\alpha_i;\alpha_{i+1}{}^{\mathsf{T}}$ is an atom. If $i$ is even, we get $\alpha_i;x^{\mathsf{T}} = (x;\alpha_i^{\mathsf{T}})^{\mathsf{T}} \subseteq S^{\mathsf{T}} \subseteq S$, since $S$ is symmetric. The point property yields $\alpha_i \subseteq S;x \subseteq S;\mathsf{L}$ and thus $\alpha_i \neq \alpha_0$, because $\alpha_0 = p \subseteq u = \overline{S;\mathsf{L}}$ by construction of $\alpha$ and Definition 8.1. Lemma A.1.(ii) implies $i \neq 0$ and that $i-1$ is odd. By Lemma A.1.(i) we get $\alpha_{i-1};\alpha_i{}^{\mathsf{T}} \subseteq S$ and, since $S$ is symmetric, also $\alpha_i;\alpha_{i-1}{}^{\mathsf{T}} \subseteq S$. So, we have $p;\alpha_{i-1}{}^{\mathsf{T}} = p;\alpha_i{}^{\mathsf{T}};\alpha_i;\alpha_{i-1}{}^{\mathsf{T}} \subseteq S;S \subseteq \mathsf{I}$, which yields $p \subseteq \mathsf{I};\alpha_{i-1} = \alpha_{i-1}$ and, hence, $p = \alpha_{i-1}$, since both are points. Finally, we get $\alpha_{i-1};\alpha_i \subseteq D$ by (A.1), but also $\alpha_{i-1};\alpha_i{}^{\mathsf{T}} = p;\alpha_i{}^{\mathsf{T}} \subseteq \overline{D}$. This contradicts the fact that $\alpha_{i-1};\alpha_i{}^{\mathsf{T}}$ is an atom.

(ii) Using the symmetry of $S$ and $D$ and a Schröder rule we get for all $i \in \mathbb{N}_{<2n+2}$ that

$$(S \cap \overline{D});\alpha_i \subseteq \mathsf{O} \iff \alpha_i{}^{\mathsf{T}};\overline{(\overline{S} \cup D)} \subseteq \mathsf{O} \iff \alpha_i;\mathsf{L} \subseteq \overline{S} \cup D ,$$

and Lemma A.3.(i) shows that $\alpha_i;\mathsf{L} \subseteq \overline{S} \cup D$. This yields $\bigcup_{j\in\mathbb{N}_{<2n+2}} \alpha_j;\mathsf{L} \subseteq \overline{S} \cup D$. We thus obtain:

$$\begin{aligned}
\mathsf{L}_{XX};\left(D \cap \overline{S}\right) &= \mathsf{L}_{XX};\left(\bigcup_{i\in\mathbb{N}_{<n+1}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}} \cup \alpha_{2i+1};\alpha_{2i}{}^{\mathsf{T}}\right) && \text{by Lemma A.2.(ii)}\\
&= \bigcup_{i\in\mathbb{N}_{<n+1}} \mathsf{L}_{XX};\alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}} \cup \mathsf{L}_{XX};\alpha_{2i+1};\alpha_{2i}{}^{\mathsf{T}}\\
&= \bigcup_{i\in\mathbb{N}_{<n+1}} \mathsf{L}_{X\mathbb{1}};\alpha_{2i+1}{}^{\mathsf{T}} \cup \mathsf{L}_{X\mathbb{1}};\alpha_{2i}{}^{\mathsf{T}} && \text{points are surjective}\\
&= \bigcup_{i\in\mathbb{N}_{<2n+2}} \mathsf{L}_{X\mathbb{1}};\alpha_i{}^{\mathsf{T}} && \text{combining indices}\\
&= \left(\bigcup_{i\in\mathbb{N}_{<2n+2}} \alpha_i;\mathsf{L}_{\mathbb{1}X}\right)^{\mathsf{T}} && \text{as } \mathsf{L}_{X\mathbb{1}}{}^{\mathsf{T}} = \mathsf{L}_{\mathbb{1}\mathbb{1}};\mathsf{L}_{X\mathbb{1}}{}^{\mathsf{T}} = \mathsf{L}_{\mathbb{1}X} \text{ by (4)}\\
&\subseteq \overline{S} \cup D && \text{see above, } S \text{ and } D \text{ are symmetric .}
\end{aligned}$$

We can now apply a Schröder rule and obtain $(S \cap \overline{D});(D \cap \overline{S}) = \overline{(\overline{S} \cup D)};(D \cap \overline{S})^{\mathsf{T}} \subseteq \mathsf{O}$, where we also use a de Morgan rule and the fact that $D, S$ are symmetric. □

After these preparations we can now prove the main result of this section. It concludes the proof of Theorem 8.1, since it implies that $S$ is not a maximum symmetric $R$-matching.

**Theorem A.1.**

*(i)* $S_+$ *is a symmetric R-matching.*

*(ii)* $|S_+| = 2 + |S|$.

*Proof.* (i) The symmetry of $S_+$ follows from the symmetry of $S$ and $D$. The inclusion $S_+ \subseteq R$ is shown as follows:

$$\begin{aligned} S_+ &= (S \cap \overline{D}) \cup (D \cap \overline{S}) && \text{definition of } S_+ \text{ by (A.1)} \\ &\subseteq R \cup (D \cap \overline{S}) && S \cap \overline{D} \subseteq S \subseteq R \\ &= R \cup \textstyle\bigcup_{i \in \mathbb{N}_{<n+1}} \alpha_{2i};\alpha_{2i+1}^{\mathsf{T}} \cup \alpha_{2i+1};\alpha_{2i}^{\mathsf{T}} && \text{by Lemma A.2.(ii)} \\ &\subseteq R \cup T && \alpha_{2i};\alpha_{2i+1}^{\mathsf{T}} \cup \alpha_{2i+1};\alpha_{2i}^{\mathsf{T}} \subseteq T \text{ by Lemma A.1.(i) and } T = T^{\mathsf{T}} \\ &= R && \text{since } T \subseteq R\,. \end{aligned}$$

It remains to show the matching property, i.e., $S_+;S_+ \subseteq \mathsf{I}$. The symmetry of $S$ and $D$, as well as Lemma A.3.(ii) yield

$$(D \cap \overline{S});(S \cap \overline{D}) = ((S \cap \overline{D})^{\mathsf{T}};(D \cap \overline{S})^{\mathsf{T}})^{\mathsf{T}} = ((S \cap \overline{D});(D \cap \overline{S}))^{\mathsf{T}} = \mathsf{O}\,. \tag{A.2}$$

We also get the following inclusion using the fact that $\alpha_{2i+1}{}^{\mathsf{T}};\alpha_{2j} = \alpha_{2i}{}^{\mathsf{T}};\alpha_{2j+1} = \mathsf{O}$, for all $i, j \in \mathbb{N}_{<n+1}$,

$$\begin{aligned} &(D \cap \overline{S});(D \cap \overline{S}) = \Big(\textstyle\bigcup_{i \in \mathbb{N}_{<n+1}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}} \cup \alpha_{2i+1};\alpha_{2i}{}^{\mathsf{T}}\Big);\Big(\bigcup_{i \in \mathbb{N}_{<n+1}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}} \cup \alpha_{2i+1};\alpha_{2i}{}^{\mathsf{T}}\Big) \\ &= \textstyle\bigcup_{i,j \in \mathbb{N}_{<n+1}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}};\alpha_{2j};\alpha_{2j+1}{}^{\mathsf{T}} \cup \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}};\alpha_{2j+1};\alpha_{2j}{}^{\mathsf{T}} \cup \alpha_{2i+1};\alpha_{2i}{}^{\mathsf{T}};\alpha_{2j};\alpha_{2j+1}{}^{\mathsf{T}} \cup \alpha_{2i+1};\alpha_{2i}{}^{\mathsf{T}};\alpha_{2j+1};\alpha_{2j}{}^{\mathsf{T}} \\ &= \textstyle\bigcup_{i,j \in \mathbb{N}_{<n+1}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}};\alpha_{2j+1};\alpha_{2j}{}^{\mathsf{T}} \cup \alpha_{2i+1};\alpha_{2i}{}^{\mathsf{T}};\alpha_{2j};\alpha_{2j+1}{}^{\mathsf{T}} && \text{see above} \\ &= \textstyle\bigcup_{i \in \mathbb{N}_{<n+1}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}};\alpha_{2j+1};\alpha_{2j}{}^{\mathsf{T}} \cup \alpha_{2i+1};\alpha_{2i}{}^{\mathsf{T}};\alpha_{2j};\alpha_{2j+1}{}^{\mathsf{T}} && \text{point property, Lemma A.1.(ii)} \\ &\subseteq \textstyle\bigcup_{i \in \mathbb{N}_{<n+1}} \alpha_{2i};\alpha_{2i}{}^{\mathsf{T}} \cup \alpha_{2i+1};\alpha_{2i+1}{}^{\mathsf{T}} && \text{point property, Lemma A.1.(ii)} \\ &\subseteq \mathsf{I}\,. && \text{since points are injective} \end{aligned}$$

Now the following calculation shows the claim:

$$\begin{aligned} S_+;S_+ &= ((S \cap \overline{D}) \cup (D \cap \overline{S}));((S \cap \overline{D}) \cup (D \cap \overline{S})) && \text{definition of } S_+ \text{ by (A.1)} \\ &\subseteq (S \cap \overline{D});(S \cap \overline{D}) \cup (D \cap \overline{S});(D \cap \overline{S}) && \text{by Lemma A.3.(ii) and (A.2)} \\ &\subseteq \mathsf{I} && \text{see above, } (S \cap \overline{D});(S \cap \overline{D}) \subseteq S;S \subseteq \mathsf{I}\,. \end{aligned}$$

(ii) We have $\bigcup_{i \in \mathbb{N}_{<n}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}} \subseteq S$ by Lemma A.1.(i) and $\bigcup_{i \in \mathbb{N}_{<n}} \alpha_{2i+1};\alpha_{2i}{}^{\mathsf{T}} = (\bigcup_{i \in \mathbb{N}_{<n}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}})^{\mathsf{T}} \subseteq S^{\mathsf{T}} = S$, since $S$ is symmetric, hence $(\bigcup_{i \in \mathbb{N}_{<n}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}}) \cup (\bigcup_{i \in \mathbb{N}_{<n}} \alpha_{2i+1};\alpha_{2i}{}^{\mathsf{T}}) \subseteq S$. Based on this inclusion, we calculate

$$\begin{aligned} |S \cap \overline{D}| &= \Big|S \cap \overline{\Big(\textstyle\bigcup_{i \in \mathbb{N}_{<n}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}}\Big) \cup \Big(\bigcup_{i \in \mathbb{N}_{<n}} \alpha_{2i+1};\alpha_{2i}{}^{\mathsf{T}}\Big)}\Big| \\ &= |S| - \Big|\Big(\textstyle\bigcup_{i \in \mathbb{N}_{<n}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}}\Big) \cup \Big(\bigcup_{i \in \mathbb{N}_{<n}} \alpha_{2i+1};\alpha_{2i}{}^{\mathsf{T}}\Big)\Big| \\ &= |S| - \Big|\textstyle\bigcup_{i \in \mathbb{N}_{<n}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}}\Big| - \Big|\bigcup_{i \in \mathbb{N}_{<n}} \alpha_{2i+1};\alpha_{2i}{}^{\mathsf{T}}\Big| + \Big|\Big(\bigcup_{i \in \mathbb{N}_{<n}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}}\Big) \cap \Big(\bigcup_{i \in \mathbb{N}_{<n}} \alpha_{2i+1};\alpha_{2i}{}^{\mathsf{T}}\Big)\Big|\,, \end{aligned}$$

using Lemma A.2.(i), then that the union is contained in $S$ and, finally, axiom (C3). The points of the sequence $\alpha$ are pairwise different due to Lemma A.1.(ii) and, thus the Dedekind rule together with the point property imply that $\alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}} \cap \alpha_{2j+1};\alpha_{2j}{}^{\mathsf{T}} = \mathsf{O}$ for all $i, j \in \mathbb{N}_{<n}$. So, axiom (C1) yields for the fourth expression

$$\Big|\Big(\textstyle\bigcup_{i \in \mathbb{N}_{<n}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}}\Big) \cap \Big(\bigcup_{i \in \mathbb{N}_{<n}} \alpha_{2i+1};\alpha_{2i}{}^{\mathsf{T}}\Big)\Big| = \Big|\bigcup_{i,j \in \mathbb{N}_{<n}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}} \cap \alpha_{2j+1};\alpha_{2j}{}^{\mathsf{T}}\Big| = |\mathsf{O}| = 0\,.$$

This shows:

$$
\begin{aligned}
\left|S \cap \overline{D}\right| &= |S| - \left|\bigcup_{i\in\mathbb{N}_{<n}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}}\right| - \left|\bigcup_{i\in\mathbb{N}_{<n}} \alpha_{2i+1};\alpha_{2i}{}^{\mathsf{T}}\right| \\
&= |S| - \left(\sum_{i\in\mathbb{N}_{<n}} \left|\alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}}\right|\right) - \left(\sum_{i\in\mathbb{N}_{<n}} \left|\alpha_{2i+1};\alpha_{2i}{}^{\mathsf{T}}\right|\right) && \text{by (C3) and (C1) as above} \\
&= |S| - \left(\sum_{i\in\mathbb{N}_{<n}} |\alpha_{2i}| \cdot |\alpha_{2i+1}|\right) - \left(\sum_{i\in\mathbb{N}_{<n}} |\alpha_{2i+1}| \cdot |\alpha_{2i}|\right) && \text{by Theorem 5.1} \\
&= |S| - \left(\sum_{i\in\mathbb{N}_{<n}} 1\right) - \left(\sum_{i\in\mathbb{N}_{<n}} 1\right) && \text{by Lemma 5.1} \\
&= |S| - 2n \,.
\end{aligned}
$$

Similarly, we compute:

$$
\begin{aligned}
\left|D \cap \overline{S}\right| &= \left|\left(\bigcup_{i\in\mathbb{N}_{<n+1}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}}\right) \cup \left(\bigcup_{i\in\mathbb{N}_{<n+1}} \alpha_{2i+1};\alpha_{2i}{}^{\mathsf{T}}\right)\right| && \text{by Lemma A.2.(ii)} \\
&= \left|\bigcup_{i\in\mathbb{N}_{<n+1}} \alpha_{2i};\alpha_{2i+1}{}^{\mathsf{T}}\right| + \left|\bigcup_{i\in\mathbb{N}_{<n+1}} \alpha_{2i+1};\alpha_{2i}{}^{\mathsf{T}}\right| && \text{same arguments as above} \\
&= \left(\sum_{i\in\mathbb{N}_{<n+1}} 1\right) + \left(\sum_{i\in\mathbb{N}_{<n+1}} 1\right) && \text{same arguments as above} \\
&= 2(n+1) \,.
\end{aligned}
$$

These auxiliary computations, finally, yield:

$$
\begin{aligned}
|S_+| &= |(S \cap \overline{D}) \cup (D \cap \overline{S})| && \text{definition of } S_+ \text{ by (A.1)} \\
&= |S \cap \overline{D}| + |D \cap \overline{S}| - |S \cap \overline{D} \cap D \cap \overline{S}| && \text{by axiom (C3)} \\
&= |S \cap \overline{D}| + |D \cap \overline{S}| && \text{by axiom (C1), as } S \cap \overline{D} \cap D \cap \overline{S} = \mathsf{O} \\
&= |S| - 2n + 2(n+1) \\
&= |S| + 2 \,.
\end{aligned}
$$

□