UNIVERSITÄT AUGSBURG

Semiring Neighbours

Peter Höfner

Report 2005-19 November 2005

INSTITUT FÜR INFORMATIK
D-86135 AUGSBURG

Semiring Neighbours

Peter Höfner

Institut für Informatik, Universität Augsburg
D-86135 Augsburg, Germany
hoefner@informatik.uni-augsburg.de

Abstract. In 1996 Zhou and Hansen proposed a first-order interval logic called *Neighbourhood Logic* (NL) for specifying liveness and fairness of computing systems and also defining notions of real analysis in terms of expanding modalities. After that, Roy and Zhou presented a sound and relatively complete Duration Calculus as an extension of NL.
We present an embedding of NL into an idempotent semiring of intervals. This embedding allows us to extend NL from single intervals to sets of intervals as well as to extend the approach to arbitrary idempotent semirings. We show that most of the required properties follow directly from Galois connections, hence we get the properties for free. As one important result we get that some of the axioms which were postulated for NL can be dropped since they are theorems in our generalisation. Furthermore, we present some possible interpretations for neighbours beyond intervals. Here we discuss for example reachability in graphs and applications to hybrid systems. At the end of the paper we add finite and infinite iteration to NL and extend idempotent semirigs to Kleene algebras and ω algebras. These extensions are useful for formulating repetitive properties and procedures like loops.

1 Introduction

Chop-based interval temporal logics, such as ITL [8] and IL [5] are useful for the specification and verification of safety properties of real-time systems. In these logics, one can easily express a lot of properties such as

"if ϕ holds for an interval, then there is a subinterval where ψ holds".

As it is shown in [18], these logics cannot express all desired properties. E.g., (unbounded) liveness properties such as

"eventually there will be an interval where ϕ holds"

is not expressible in these logics. Surprisingly, these logics cannot even express state transitions. That is why in Chapter 9 of [18] extra atomic formulas are introduced. As it is shown there, the reason is that the modality *chop* $\frown$ is a *contracting* modality, in the sense that the truth value of $\phi\frown\psi$ on $[b,e]$ only depends on subintervals of $[b,e]$:

$\phi\frown\psi$ holds on $[b,e]$ iff
there exists $m \in [b,e]$ such that ϕ holds on $[b,m]$ and ψ holds on $[m,e]$.

Sometimes, e.g. in [1], contracting operators are also called *constructing.* Hence Zhou and Hansen proposed a first-order interval logic called *Neighbourhood Logic* (NL) in 1996 [17, 16]. This first-order logic was proposed for specifying liveness and fairness of computing systems and also defining notions of real analysis in terms of expanding modalities. After that, in 1997, Roy and Zhou presented a sound and relatively complete Duration Calculus as an extension of NL [14]. They had already shown that the basic unary interval modalities of [7] and the three binary interval modalities (C, T and D) of [15] could be defined in NL.

In this paper, we present an embedding of NL into the semiring of intervals presented e.g. in [10]. This embedding allows us to extend NL from single intervals to sets of intervals as well as to extend the approach to arbitrary idempotent semirings. Because of work done in [17] it is also an extension of [7] and [15]. In Section 4 we show that most of the required properties follow directly from Galois connections, hence we get the properties for free. As one important result we get that some of the axioms which were postulated for NL can be dropped since they are theorems in our generalisation. In Section 5 we briefly present some possible interpretations of neighbours in other models. Here we discuss for example reachability in graphs and applications for hybrid systems. At the end of the paper we add finite and infinite iteration to NL and extend idempotent semirigs to Kleene algebras and ω algebras. These extensions are useful for formulating repetitive properties and procedures like loops in progams.

2 About Neighbourhood Logic

In [17] Zhou and Hansen introduce *left* and *right neighbourhoods* as primitive intervals to define other unary and binary modalities of intervals in a first-order logic. For this, we need intervals as carrier sets. That is why we define *intervals* over a poset of *timepoints* in the usual way as

$$[b, e] \stackrel{\text{def}}{=} \{x \mid b \le x \le e\} \ , \ \text{where } b \le e \ ,$$

$b, e, x \in \mathbb{T}\text{ime}$ and $(\mathbb{T}\text{ime}, +, 0)$ is a commutative monoid. Note that we only consider non-empty intervals. Furthermore, we postulate a subtraction $-$ on $\mathbb{T}$ime satisfying for any interval $[b, e]$ the equations $e - b \ge 0$ and $e - b = 0 \Leftrightarrow e = b$. Hence, it is possible to calculate the *length* l of the interval $[b, e]$ as $e - b$. Additionally, $\mathbb{T}$ime has to be cancellative with respect to $+$, i.e., $a+c = b+c \Rightarrow a = b$ and $c+a = c+b \Rightarrow a = b$. E.g. one can use $\mathrm{I\!R}$, the set of real numbers, as $\mathbb{T}$ime.
The two simple expanding modalities $\Diamond_l\phi$ and $\Diamond_r\phi$ are defined as follows:

$\Diamond_r\phi$ holds on $[b, e]$ iff there exists $\delta \ge 0$ such that ϕ holds on $[e, e + \delta]$,
$\Diamond_l\phi$ holds on $[b, e]$ iff there exists $\delta \ge 0$ such that ϕ holds on $[b - \delta, b]$,

or, by setting $a \stackrel{\text{def}}{=} b - \delta$ and $c \stackrel{\text{def}}{=} e + \delta$,

$\Diamond_r\phi$ holds on $[b, e]$ iff there exists $c \ge e$ such that ϕ holds on $[e, c]$,
$\Diamond_l\phi$ holds on $[b, e]$ iff there exists $a \le b$ such that ϕ holds on $[a, b]$.

where ϕ is a *formula* of NL, that is either true or false. More precisely, the set of *terms* $\theta, \theta_i \in Term$ is defined by the abstract syntax [18]

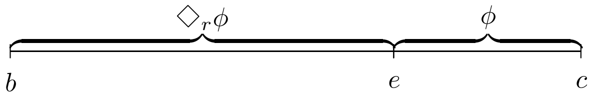
$$\theta ::= x|v|f^n(\theta_1, \ldots, \theta_n)$$

and the set of *formulas* of NL by

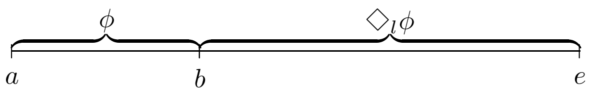
$$\phi ::= X|G^n(\theta_1, \ldots, \theta_n)|\neg\phi|\phi \vee \psi|(\exists\ x)\psi|\Diamond_l\phi|\Diamond_r\phi\ ,$$

where x is a *global variable*, v is a *temporal variable*, f is a *global function symbol*, G a *global relation symbol* and X a *temporal propositional letter* (a true-valued interval function). More details can be found in [18].

With $\Diamond_r(\Diamond_l)$ one can reach the left (right) neighbourhood of the beginning (ending) point of an interval:



where $c = e + \delta$



where $a = b - \delta$

In contrast to the chop operator the neighbourhood modalities are *expanding* modalities, i.e., they are not contracting operators. Thus $\Diamond_l$ and $\Diamond_r$ depends not only on subintervals of an interval $[b, e]$, but also on intervals "outside". In [17] it is shown that the modalities of [7] and [15] as well as the chop operator can be expressed by the neighbourhood modalities.

3 Embedding Neighbourhood Logic into semirings

3.1 Basic definitions

First, we repeat the basic definitions of semirings and related algebraic structures and operators. More details about semirings, domain semirings, etc. can be found in [9, 4, 6].

A *semiring* is a quintuple $(S, +, \cdot, 0, 1)$ such that $(S, +, 0)$ is a commutative monoid and $(S, \cdot, 1)$ is a monoid such that $\cdot$ is distributive over $+$ and *strict*, i.e., $0 \cdot a = 0 = a \cdot 0$. The semiring is *idempotent* if $+$ is, i.e. $a + a = a$. On idempotent semirings the relation $a \leq b \overset{\text{def}}{\Leftrightarrow} a + b = b$ is a partial order, called the *natural order* on S. The definition implies that 0 is the least element and $+$ and $\cdot$ are isotone with respect to $\leq$. If S has a greatest element, we denote it by $\top$. An idempotent semiring S is called a *quantale* if S is a complete lattice under the natural order and $\cdot$ is universally disjunctive. Following Conway [2] one might also call a quantale a *standard Kleene algebra*. An important semiring (that is even a quantale) is REL, the algebra of binary relations over a set under relational composition.

A *test semiring (quantale)* is a pair $(S, \mathsf{test}(S))$, where S is an idempotent semiring (a quantale) and $\mathsf{test}(S) \subseteq [0, 1]$ is a Boolean subalgebra of the interval

$[0,1]$ of S such that $0, 1 \in \mathsf{test}(S)$ and join and meet in $\mathsf{test}(S)$ coincide with $+$ and $\cdot$. This definition corresponds to the one in [12]. We will use $a, b, c \ldots$ and x, y, z for arbitrary S-elements and $p, q, r, \ldots$ for tests. By $\neg$ we denote complementation in $\mathsf{test}(S)$.

A *domain semiring (quantale)* is a pair $(S, \ulcorner)$, where S is a test semiring (quantale) and the *domain* operation $\ulcorner : S \to \mathsf{test}(S)$ satisfies

$$a \leq \ulcorner a \cdot a \quad \text{(d1)}, \qquad\qquad \ulcorner(p \cdot a) \leq p \quad \text{(d2)}.$$

The relevant consequences of $\ulcorner$ are shown in [4]. To further explain (d1) and (d2) we note that their conjunction is equivalent to each of

$$\ulcorner a \leq p \Leftrightarrow \neg p \cdot a \leq 0 \ , \tag{gla}$$

$$\ulcorner a \leq p \Leftrightarrow a \leq p \cdot a \ , \tag{llp}$$

which constitute elimination laws for domain. (gla) says that $\neg p \cdot a$ is the greatest left annihilator of a. (llp) says that $p \cdot a$ is the least left preserver of a. Moreover, domain is universally disjunctive and hence strict, i.e., $\ulcorner 0 = 0$. Furthermore we can strengthen (d1) to the equation

$$a = \ulcorner a \cdot a \ . \tag{d1'}$$

$\ulcorner$ need not be to exist on every test semiring, but in the case of quantales domain is guaranteed to exist. A corresponding codomain operation $\urcorner : S \to \mathsf{test}(S)$ can be defined analogously. It can be seen as the domain operation in the opposite semiring, where opposition just changes the order of multiplication. S is called a *bidomain* semiring (quantale) if there are domain and codomain operations. In bidomain semirings we have the following separability

$$a\urcorner \cdot \ulcorner b \leq 0 \Leftrightarrow a\urcorner \cdot b \leq 0 \Leftrightarrow a \cdot \ulcorner b \leq 0 \ . \tag{sep}$$

Proof. By shunting and (gla) we get $a\urcorner \cdot \ulcorner b \leq 0 \Leftrightarrow \ulcorner b \leq \neg a\urcorner \Leftrightarrow a\urcorner \cdot b \leq 0$. The second equivalence is by shunting and a law for $\urcorner$ analogously to (gla). □

In [10] we showed that the structure $\mathrm{INT} = (\mathcal{P}(\mathbb{I}), \cup, ;, \emptyset, \mathbb{1})$ is a Boolean quantale, where $\mathbb{I} \stackrel{\mathrm{def}}{=} \{[b,e] \mid b \leq e, b, e \in \mathbb{T}\mathrm{ime}\}$ is the set of all intervals, $; : \mathcal{P}(\mathbb{I}) \times \mathcal{P}(\mathbb{I}) \to \mathcal{P}(\mathbb{I})$ defines the elementwise interval composition and $\mathbb{1} \stackrel{\mathrm{def}}{=} \{[b,b] \mid b \in \mathbb{T}\mathrm{ime}\}$ is the neutral element w.r.t. multiplication. The definition of interval composition says that $[a,b] \,;\, [c,d]$ is defined if and only if $b = c$, i.e., the interval $[c,d]$ is part of the "right neighbourhood" of $[a,b]$, or, symmetrically, iff $[a,b]$ is part of the "left neighbourhood" of $[c,d]$. Here the domain (codomain) characterises the starting points (end points) of intervals, i.e., for $x \in \mathcal{P}(\mathbb{I})$

$$\ulcorner x = \{[b,b] : [b,e] \in x\} \qquad \text{and} \qquad x\urcorner = \{[e,e] : [b,e] \in x\} \ .$$

In any quantale like INT the *left residual* a/b and the *right residual* $a \backslash b$ exist and are characterised by the Galois connections

$$x \leq a/b \stackrel{\mathrm{def}}{\Leftrightarrow} x \cdot b \leq a \quad \text{and} \quad x \leq a \backslash b \stackrel{\mathrm{def}}{\Leftrightarrow} a \cdot x \leq b.$$

In INT the first of these operations are characterised pointwise by $t \in V/U \Leftrightarrow \forall\ u \in U : t\,;u \in V$ (provided $t\,;u$ is defined). Based on the left and right residuals, in a Boolean quantale the *right detachment* $a\lfloor b$ and the *left detachment* $a\rfloor b$ can be defined as

$$a\lfloor b \stackrel{\text{def}}{=} \overline{\overline{a}/b} \quad \text{and} \quad a\rfloor b \stackrel{\text{def}}{=} \overline{a\backslash\overline{b}}$$

The pointwise characterisation of right detachment in INT is

$$t \in V\lfloor U \Leftrightarrow \exists\ u \in U : t\,;u \in V\ .$$

By de Morgan's laws the Galois connection for / transforms into the exchange law $a\lfloor b \leq x \Leftrightarrow \overline{x}\cdot b \leq \overline{a}$ that generalises the Schröder rule of relational calculus. More details concerning residuals and detachments can be found in [13].

3.2 From detachments and domain to neighbourhoods

Now let us have a look at the special case where $V = \top \stackrel{\text{def}}{=} \mathbb{I}$ (the set of all intervals) and $U = \mathbb{I}_\phi \stackrel{\text{def}}{=} \{[b,e]\,|\,[b,e] \in \mathbb{I}, \phi \text{ holds on } [b,e]\}$ (the set of all intervals where ϕ holds). An example which is often used by Zhou et al. is the set of all intervals with length $l > 0$. In our notation this is the same as $\mathbb{I}_{l>0} = \overline{\mathbb{1}}$. Now we can calculate an algebraic expression for the right neighbourhood $\Diamond_r\phi$.

$$\begin{aligned}
\Diamond_r\phi \text{ holds on } [b,e] &\Leftrightarrow \exists\ [e,u_2] \in \mathbb{I} \text{ such that } \phi \text{ holds on } [e,u_2] \\
&\Leftrightarrow \exists\ [e,u_2] \in \mathbb{I}_\phi \\
&\Leftrightarrow \exists\ [u_1,u_2] \in \mathbb{I}_\phi : u_1 = e \\
&\Leftrightarrow \exists\ [u_1,u_2] \in \mathbb{I}_\phi : [b,e]\,;[u_1,u_2] \text{ is defined} \\
&\Leftrightarrow \exists\ [u_1,u_2] \in \mathbb{I}_\phi : [b,e]\,;[u_1,u_2] \in \top \\
&\Leftrightarrow [b,e] \in \top\lfloor\mathbb{I}_\phi\ .
\end{aligned}$$

Similarly, we get

$$\Diamond_l\phi \text{ holds on } [b,e] \Leftrightarrow [b,e] \in \mathbb{I}_\phi\rfloor\top\ .$$

Hence , in a quantale, we can generalise the neighbourhood modalities to sets of intervals by setting

$$\begin{aligned}
\Diamond_r\phi \text{ holds on } x \in \mathcal{P}(\mathbb{I}) &\Leftrightarrow x \leq \top\lfloor\mathbb{I}_\phi\ , \\
\Diamond_l\phi \text{ holds on } x \in \mathcal{P}(\mathbb{I}) &\Leftrightarrow x \leq \mathbb{I}_\phi\rfloor\top\ .
\end{aligned}$$

All results given by Zhou, Hansen and Roy can be adapted to the semiring of intervals INT.

On the other hand we know that INT forms also a bidomain semiring. As we showed above, the domain (codomain) characterises the starting points (ending points) of intervals. This implies another view of the neighbourhood modalities.

$$\begin{aligned}
\Diamond_r\phi \text{ holds on } \{[b,e]\} &\Leftrightarrow \exists\ [u_1,u_2] \in \mathbb{I}_\phi : [b,e]\,;[u_1,u_2] \text{ is defined} \\
&\Leftrightarrow \exists\ [u_1,u_2] \in \mathbb{I}_\phi : e = u_1 \\
&\Leftrightarrow \{[b,e]\}^{\urcorner} \leq {}^{\ulcorner}\mathbb{I}_\phi,
\end{aligned}$$

In general we get an alternative definition of $\Diamond_l\phi$ and $\Diamond_r\phi$.

$$\Diamond_r\phi \text{ holds on } x \in \mathcal{P}(\mathbb{I}) \;\Leftrightarrow\; x^{\urcorner} \leq {}^{\ulcorner}\mathbb{I}_\phi \ ,$$
$$\Diamond_l\phi \text{ holds on } x \in \mathcal{P}(\mathbb{I}) \;\Leftrightarrow\; {}^{\ulcorner}x \leq \mathbb{I}_\phi^{\urcorner} \ .$$

and get the equivalences $x \leq \mathbb{I}_\phi\rfloor\top \;\Leftrightarrow\; {}^{\ulcorner}x \leq \mathbb{I}_\phi^{\urcorner}$ and $x \leq \top\lfloor\mathbb{I}_\phi \;\Leftrightarrow\; x^{\urcorner} \leq {}^{\ulcorner}\mathbb{I}_\phi$ in INT. This relation is a general property between detachments and domain in any detachment semiring when the semiring is *modal* like INT, i.e., satisfies ${}^{\ulcorner}(a \cdot {}^{\ulcorner}b) \;=\; {}^{\ulcorner}(a \cdot b)$.

Lemma 3.1 *The following properties are equivalent:*

$(i) \quad x \cdot p \leq 0 \qquad\qquad (ii) \quad x \leq x \cdot \neg p \qquad\qquad (iii) \quad x \leq \top \cdot \neg p$

Proof.

(i) $\Rightarrow$ (ii)	$x \;=\; x \cdot (p + \neg p) x \cdot p + x \cdot \neg p \leq 0 x \cdot \neg p \;=\; x \cdot \neg p$
(ii) $\Rightarrow$ (iii)	By isotonicity we get $x \leq x \cdot \neg p \leq \top \cdot \neg p$.
(iii) $\Rightarrow$ (i)	$x \cdot p \leq \top \neg p \cdot p \;=\; \top \cdot 0 \;=\; 0$

□

Lemma 3.2 *If S forms a detachment semiring as well as a bidomain semiring and has a greatest element $\top$, then*

(i) $\top\lfloor y \leq \top\lfloor{}^{\ulcorner}y \;=\; \top \cdot {}^{\ulcorner}y$ *and* $y\rfloor\top \leq y^{\urcorner}\rfloor\top \;=\; y^{\urcorner} \cdot \top$,
(ii) $x \leq \top\lfloor y \;\Rightarrow\; x^{\urcorner} \leq {}^{\ulcorner}y$ *and* $x \leq y\rfloor\top \;\Rightarrow\; {}^{\ulcorner}x \leq y^{\urcorner}$,
(iii) if S is modal, we get equations in (i) and equivalences in (ii).

Proof.

(i) We use the principle of indirect inequality.

$$\begin{array}{ll} & \top\lfloor y \leq w \\ \Leftrightarrow & \quad \{\!\![\text{ exchange }]\!\!\} \\ & \overline{w} \cdot y \leq 0 \\ \Leftarrow & \quad \{\!\![\text{ isotonicity and (d1')}]\!\!\} \\ & \overline{w} \cdot {}^{\ulcorner}y \leq 0 \\ \Leftrightarrow & \quad \{\!\![\text{ exchange }]\!\!\} \\ & \top\lfloor{}^{\ulcorner}y \leq w \end{array}$$

For the second assertion let $p \in \mathsf{test}(S)$. First we show $0/p \;=\; \top \cdot \neg p$. By definition and Lemma 3.1 we get

$$x \leq 0/p \Leftrightarrow x \cdot p \leq 0 \Leftrightarrow x \leq \top \cdot \neg p \ .$$

Hence we have by definition and $\overline{\top \cdot p} \;=\; \top \cdot \neg p$

$$\top\lfloor p \;=\; \overline{0/p} \;=\; \overline{\top \cdot \neg p} \;=\; \top \cdot p \ .$$

(ii) $\quad x \leq \top\lfloor y$

$\Rightarrow \quad \{\![\text{(i)}]\!\}$

$x \leq \top \cdot \ulcorner y$

$\Leftrightarrow \quad \{\![\text{Lemma 3.1}]\!\}$

$x \leq x \cdot \ulcorner y$

$\Leftrightarrow \quad \{\![\text{(lrp)}]\!\}$

$x\urcorner \leq \ulcorner y$

(iii) If S is modal, we have $a \cdot b \leq 0 \Leftrightarrow a \cdot \ulcorner b \leq 0$ (see e.g. Lemma 5.7 in [4]) and therefore the second step in the proof of (i) and the first step of (ii) become equivalences.

□

As a first result we note that at least one of the eight axioms, which are claimed in [17] can be dropped and is in fact a theorem in domain semirings. More simplifications on calculations are given in Section 4.1 after introducing a more general framework of neighbourhoods.

Lemma 3.3 $\Diamond(\phi \vee \psi) \Leftrightarrow \Diamond\phi \vee \Diamond\psi$*, where* $\Diamond$ *is either* $\Diamond_r$ *or* $\Diamond_l$*.*

Hence Axiom 4 of [17] is a conclusion.

In Section 4 we will give the proof in a more general environment (see Lemma 4.7). Now we will discuss the box operators $\Box_l\phi \stackrel{\text{def}}{=} \sim\Diamond_l\sim\phi$ and $\Box_r \stackrel{\text{def}}{=} \sim\Diamond_l\sim\phi$ of Zhou and Hansen in detachment and bidomain semirings, respectively. Here, $\sim$ is the negation of truth values, i.e., $\sim(\mathsf{true}) = \mathsf{false}$ and $\sim(\mathsf{false}) = \mathsf{true}$. In [16, 18, 17] it is denoted as usual by $\neg$. But this symbol clashes with the negation symbol of tests. The meaning of $\Box_l\phi$ and $\Box_r\phi$ is the following:

$$\Box_r\phi \text{ holds on } [b,e] \text{ iff } \phi \text{ holds on all right neighbours of } [b,e]\ ,$$
$$\Box_l\phi \text{ holds on } [b,e] \text{ iff } \phi \text{ holds on all left neighbours of } [b,e]\ .$$

We start again with the pointwise characterisation of $\Box$:

$$\begin{array}{rl}\Box_r\phi \text{ holds on } [b,e] & \Leftrightarrow \sim\Diamond_r\sim\phi \text{ holds on } [b,e] \\ & \Leftrightarrow \sim([b,e] \in \top\lfloor\mathbb{I}_{\sim\phi}) \\ & \Leftrightarrow [b,e] \notin \top\lfloor\mathbb{I}_{\sim\phi} \\ & \Leftrightarrow [b,e] \in \overline{\top\lfloor\mathbb{I}_{\sim\phi}} \\ & \Leftrightarrow [b,e] \in \overline{\overline{0}\lfloor\mathbb{I}_{\sim\phi}} \\ & \Leftrightarrow [b,e] \in 0/\mathbb{I}_{\sim\phi}\ ,\end{array}$$

where $\mathbb{I}_{\sim\phi} = \overline{I_\phi}$. Using the same generalisation as above we set

$$\Box_r\phi \text{ holds on } x \in \mathcal{P}(\mathbb{I}) \Leftrightarrow x \leq 0/\mathbb{I}_{\sim\phi} \Leftrightarrow x\,;\mathbb{I}_{\sim\phi} \leq 0\ ,$$
$$\Box_l\phi \text{ holds on } x \in \mathcal{P}(\mathbb{I}) \Leftrightarrow x \leq \mathbb{I}_{\sim\phi}\backslash 0 \Leftrightarrow \mathbb{I}_{\sim\phi}\,;x \leq 0\ .$$

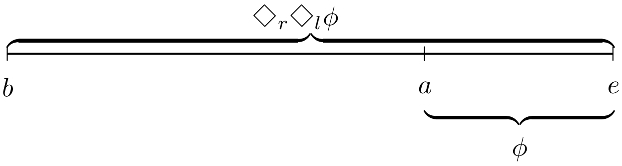
The (co-)domain view gives the following definition:

$$\begin{aligned} \Box_l\phi \text{ holds on } x \in \mathcal{P}(\mathbb{I}) &\Leftrightarrow (\mathbb{I}_{\sim\phi})^{\urcorner} \,;\, {}^{\ulcorner}x \le 0 \ , \\ \Box_r\phi \text{ holds on } x \in \mathcal{P}(\mathbb{I}) &\Leftrightarrow x^{\urcorner} \,;\, {}^{\ulcorner}(\mathbb{I}_{\sim\phi}) \le 0 \ . \end{aligned}$$

The equivalence in modal detachment semirings between the two settings of $\Box_l$ ($\Box_r$) is immediate by definition of modality and (sep).

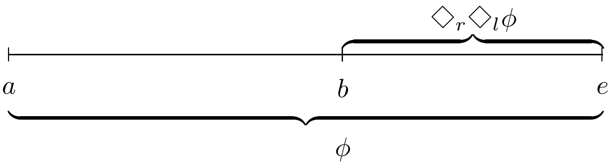
However since it is more comfortable to calculate with (co-)domain instead of detachment we will use the bidomain interpretation in the remainder. Furthermore, this interpretation is more general because we do not use residuals and detachments and therefore do not need to assume their existence.

In [17] the authors introduce the composed neighbourhood modalities $\Diamond_r\Diamond_l\phi$ and $\Diamond_l\Diamond_r\phi$ and call them *converses*. But these are very unhandy in calculations and we show that they are again diamonds closely related to $\Diamond_l$ and $\Diamond_r$.
First we want to illustrate the meaning of $\Diamond_r\Diamond_l\phi$.



where $a = e - \delta$

Here, $[a, e]$ is a postfix of $[b, e]$. But, one should mention that it is also possible that $a \le b$, i.e., $[b, e]$ is a postfix of $[a, e]$



where $a = e - \delta$

Now we have a look at $\Diamond_r\Diamond_l\phi$ using domain and codomain.

$$\begin{aligned} \Diamond_r\Diamond_l\phi \text{ holds on } x &\Leftrightarrow x^{\urcorner} \le {}^{\ulcorner}\!\left(\mathbb{I}_{\Diamond_l\phi}\right) \\ &\Leftrightarrow x^{\urcorner} \le {}^{\ulcorner}\{[b,e] \mid {}^{\ulcorner}\{[b,e]\} \le \mathbb{I}_{\phi}^{\urcorner}\} \\ &\Leftrightarrow x^{\urcorner} \le \{[b,b] \mid [b,b] \in \mathbb{I}_{\phi}^{\urcorner}\} \\ &\Leftrightarrow x^{\urcorner} \le \mathbb{I}_{\phi}^{\urcorner} \ , \\ \Diamond_l\Diamond_r\phi \text{ holds on } x &\Leftrightarrow {}^{\ulcorner}x \le {}^{\ulcorner}\mathbb{I}_{\phi} \ . \end{aligned}$$

We see that $\Diamond_r\Diamond_l\phi$ and $\Diamond_l\Diamond_r\phi$ are no more complicated than our characterisations of the single neighbourhood modalities. Thus the four neighbourhood operators ($\Diamond_l$, $\Diamond_r$, $\Diamond_l\Diamond_r$, $\Diamond_r\Diamond_l$) represent all combinations for comparing domain and codomain and therefore motivate the generalised definition in the next section.

4 Generalised Neighbourhoods and some Properties

Starting with the definitions of neighbourhoods given in Section 3 and motivated by NL we give general definitions which work on bidomain semirings. These semirings need not be quantales or Boolean, because we will not use detachments anymore. The generalisation is much more than the generalisation given in Chapter 11 of [18], where Zhou and Hansen relax intervals over $\mathbb{R}$ to intervals over $\mathbb{T}\mathrm{ime}$, where $\mathbb{T}\mathrm{ime}$ is a general definition of timepoints satisfying some axioms. We used their generalisation from the beginning.

Definition 4.1 Let S be a bidomain semiring and $x, y \in S$. Then
(i) x is a *left neighbour* of y (or $x \leq \Diamond^{n}_{l} y$ for short) iff $x^{\urcorner} \leq {}^{\ulcorner}y$,
(ii) x is a *right neighbour* of y (or $x \leq \Diamond^{n}_{r} y$ for short) iff ${}^{\ulcorner}x \leq y^{\urcorner}$,
(iii) x is a *left boundary* of y (or $x \leq \Diamond^{b}_{l} y$ for short) iff ${}^{\ulcorner}x \leq {}^{\ulcorner}y$,
(iv) x is a *right boundary* of y (or $x \leq \Diamond^{b}_{r} y$ for short) iff $x^{\urcorner} \leq y^{\urcorner}$.

We will see below that the notation using $\leq$ is justified. Now we have a closer look at the definition and its interpretation in INT. For example 4.1.(i) describes the situation, where for each element $[a, b]$ of x there exists at least one interval in y with starting point b. Hence $\Diamond_r \phi$ holds on x if and only if x is a left neighbour of $\mathbb{I}_\phi$ ($x \leq \Diamond^{n}_{l} \mathbb{I}_\phi$). The change in direction (left, right) follows from the point of view. $\Diamond_r \phi$ starts with an interval of x and has a look at elements of $\mathbb{I}_\phi$ at its right which satisfies ϕ. Whereas our definitions start at $\mathbb{I}_\phi$ and look at all intervals which are composeable from the left to an interval where ϕ holds.
In Definition 4.1 we do not postulate modality of S, which we used when motivating and deriving the formulas in Section 3. Hence we get more general calculations. Of course we cannot use the equivalences to detachment semirings given in Lemma 3.2. Starting at our definitions of neighbours and boundaries we calculate an explicit form of these operations.

Lemma 4.2 *Neighbours and boundaries can be expressed explicitly by*

$$\Diamond^{n}_{l} y = \top \cdot {}^{\ulcorner}y\ , \qquad \Diamond^{n}_{r} y = y^{\urcorner} \cdot \top\ ,$$

$$\Diamond^{b}_{l} y = {}^{\ulcorner}y \cdot \top\ , \qquad \Diamond^{b}_{r} y = \top \cdot y^{\urcorner}\ .$$

Proof. We only show the first case. The other equalities are similar.

$$x \leq \Diamond^{n}_{l} y$$
$\Leftrightarrow$ {[definition]}
$$x^{\urcorner} \leq {}^{\ulcorner}y$$
$\Leftrightarrow$ {[(llp) in the opposite semiring]}
$$x \cdot x^{\urcorner} \leq x \cdot {}^{\ulcorner}y$$
$\Leftrightarrow$ {[Lemma 3.1]}
$$x \leq \top \cdot {}^{\ulcorner}y$$

$\square$

In the case where we have a complement function on S we define perfect neighbours and boundaries. Here a complement function $\overline{} : S \to S$ has to satisfy the following three equations

$$\overline{\overline{a}} = a \quad (1), \qquad \overline{a} + a = \top \quad (2), \qquad a \leq b \Leftrightarrow \overline{b} \leq \overline{a} \quad (3),$$

where $\top$ is the greatest element which we assume to exist. We call a semiring with $\overline{}$ a *complement* semiring. Note that complement semirings form a larger class than Boolean algebras even if we define meet by $x \sqcap y \stackrel{\text{def}}{=} \overline{\overline{x} + \overline{y}}$. The reason is that we do not postulate the distributivity laws for join and meet.

Definition 4.3 Let S be a complement bidomain semiring and $x, y \in S$.
(i) x is a *perfect left neighbour* of y (or $x \leq \boxed{\eta}_l y$ for short) iff $x^{\urcorner} \cdot {}^{\ulcorner}\overline{y} \leq 0$,
(ii) x is a *perfect right neighbour* of y (or $x \leq \boxed{\eta}_r y$ for short) iff $\overline{y}^{\urcorner} \cdot {}^{\ulcorner}x \leq 0$,
(iii) x is a *perfect left boundary* of y (or $x \leq \boxed{\beta}_l y$ for short) iff ${}^{\ulcorner}x \cdot {}^{\ulcorner}\overline{y} \leq 0$,
(iv) x is a *perfect right boundary* of y (or $x \leq \boxed{\beta}_r y$ for short) iff $x^{\urcorner} \cdot \overline{y}^{\urcorner} \leq 0$.

By (iii) and (iv) we have an additional extension of NL. These two definitions define "box-operators" for the converses of neighbourhood modalities, which are not defined in the semantics of NL given in [18]. To justify the definitions above we have

Lemma 4.4 *Each perfect neighbour (boundary) is a neighbour (boundary):*

$$\boxed{\eta}_l y \leq \langle\eta\rangle_l y \ , \qquad \boxed{\eta}_r y \leq \langle\eta\rangle_r y \ , \qquad \boxed{\beta}_l y \leq \langle\beta\rangle_l y \ , \qquad \boxed{\beta}_r y \leq \langle\beta\rangle_r y \ .$$

Proof. First we get by $1 = {}^{\ulcorner}\top = {}^{\ulcorner}(a + \overline{a}) = {}^{\ulcorner}a + {}^{\ulcorner}\overline{a}$ and by shunting $\neg {}^{\ulcorner}a \leq {}^{\ulcorner}\overline{a}$.

$$
\begin{array}{ll}
 & x \leq \boxed{\eta}_l y \\
\Leftrightarrow & \quad \{\!\![\text{ definition }]\!\!\} \\
 & x^{\urcorner} \cdot {}^{\ulcorner}\overline{y} \leq 0 \\
\Leftrightarrow & \quad \{\!\![\text{ shunting }]\!\!\} \\
 & x^{\urcorner} \leq \neg {}^{\ulcorner}\overline{y} \\
\Rightarrow & \quad \{\!\![\text{ calculations above }]\!\!\} \\
 & x^{\urcorner} \leq {}^{\ulcorner}(\overline{\overline{y}}) = {}^{\ulcorner}y \\
\Leftrightarrow & \quad \{\!\![\text{ definition }]\!\!\} \\
 & x \leq \langle\eta\rangle_l y
\end{array}
$$

$\square$

We are able to characterise the box operations like neighbours/boundaries in an explicit form.

Lemma 4.5 *Perfect neighbours and perfect boundaries have the following explicit forms:*

$$\boxed{\scriptstyle\eta}_l\, y \;=\; \top \cdot \neg\, \ulcorner y \;, \qquad\qquad \boxed{\scriptstyle\eta}_r\, y \;=\; \neg\, y^{\urcorner} \cdot \top \;,$$

$$\boxed{\scriptstyle\beta}_l\, y \;=\; \neg\, \ulcorner y \cdot \top \;, \qquad\qquad \boxed{\scriptstyle\beta}_r\, y \;=\; \top \cdot \neg\, y^{\urcorner} \;.$$

Proof. Again, we only show the first equation.

$$x \le \boxed{\scriptstyle\eta}_l\, y \;\Leftrightarrow\; x^{\urcorner} \cdot \ulcorner y \le 0 \;\Leftrightarrow\; x^{\urcorner} \le \neg\, \ulcorner y \;\Leftrightarrow\; x \le \top \cdot \neg\, \ulcorner y$$

The second step uses shunting, the third step Lemma 3.1. □

To reduce calculations we introduce $\Diamond$ and $\boxdot$ as parameterised versions that can be instantiated by either $\Diamond^{\eta}_l$, $\Diamond^{\eta}_r$, $\Diamond^{\beta}_l$ or $\Diamond^{\beta}_r$ and $\boxed{\scriptstyle\eta}_l$, $\boxed{\scriptstyle\eta}_r$, $\boxed{\scriptstyle\beta}_l$ or $\boxed{\scriptstyle\beta}_r$, respectively. The instantiation must be consistent for all occurrences of $\Diamond$ and $\boxdot$. The following proofs are only done for one instance of $\Diamond$ or $\boxdot$. All other instances can be calculated in a similar way. If the "direction" of $\Diamond$ or $\boxdot$ is important we use formulas like $\Diamond_l$ and $\boxdot_r$ where only one degree of freedom remains. The above explicit form shows that boxes and diamonds are connected via the de Morgan dualities

$$\boxdot y \;=\; \overline{\Diamond \overline{y}} \qquad \text{and} \qquad \Diamond y \;=\; \overline{\boxdot \overline{y}} \;;$$

hence they form proper modal operators. Additionally we show that diamonds and boxes are lower and upper adjoints of Galois connections:

Lemma 4.6

$$\Diamond_l x \le y \;\Leftrightarrow\; x \le \boxdot_r y \;, \qquad\qquad \Diamond_r x \le y \;\Leftrightarrow\; x \le \boxdot_l y \;.$$

Proof. $\Diamond^{\eta}_l x \le y$

$\Leftrightarrow$ {[de Morgan duality]}

$\overline{\boxed{\scriptstyle\eta}_l\, \overline{x}} \le y$

$\Leftrightarrow$ {[complement law (3)]}

$\overline{y} \le \boxed{\scriptstyle\eta}_l\, \overline{x}$

$\Leftrightarrow$ {[definition of $\boxed{\scriptstyle\eta}_l$ and (1)]}

$\overline{y}^{\urcorner} \cdot \ulcorner x$

$\Leftrightarrow$ {[definition of $\boxed{\scriptstyle\eta}_r$]}

$x \le \boxed{\scriptstyle\eta}_r\, y$

□

Looking at the proof, we observe that for perfect neighbours we get the exchange rule

$$x \le \boxed{\scriptstyle\eta}_l\, y \;\Leftrightarrow\; \overline{y} \le \boxed{\scriptstyle\eta}_r\, \overline{x} \;.$$

4.1 Simplifications of Neighbourhood Logic

Since Galois connections are useful as theorem generators and dualities as theorem transformers we get many properties of (perfect) neighbours and (perfect) boundaries for free. For example we have, with $x \sqcap y = \overline{\overline{x} + \overline{y}}$,

Corollary 4.7

(i) $\Diamond$ *and* $\boxdot$ *are isotone.*
(ii) $\Diamond$ *is disjunctive and* $\boxdot$ *is conjunctive, i.e.,*

$$\Diamond(x+y) = \Diamond x + \Diamond y \ , \qquad \boxdot(x \sqcap y) = \boxdot x \sqcap \boxdot y \ .$$

(iii) We also have the cancellative laws

$$\Diamond_l \boxdot_r x \le x \le \boxdot_r \Diamond_l x \ , \qquad \Diamond_r \boxdot_l x \le x \le \boxdot_l \Diamond_r x \ .$$

With Lemma 4.7.(ii) we have now proved the claim given in Lemma 3.3. So at least one axiom of the Neighbourhood Logic of Zhou and Hansen is a theorem in the generalised form of bidomain semirings.
Since 0 is the least element with respect to $\le$ and domain and codomain are strict, 0 is a neighbour and boundary of each element. Furthermore special neighbours and boundaries are summarised in

Lemma 4.8

(i) $\Diamond 1 = \Diamond \top = \boxdot \top = \top \ , \quad \Diamond 0 = \boxdot 0 = 0 \ .$
(ii) $\Diamond x \le 0 \Leftrightarrow x \le 0 \ .$
(iii) By isotonicity we get $\ulcorner x \le \Diamond_l x$ *and* $x\urcorner \le \Diamond_r x$*. Additionally, we have that* x *is a left (right) boundary of itself, i.e.,* $x \le \Diamond^{\beta}_l x$ *and* $x \le \Diamond^{\beta}_r x$*.*
(iv) By the Galois connections and (i) we get $\top \le \boxdot y \Leftrightarrow \top \le y$*.*

Lemma 4.8.(iii) cannot be translated from $\Diamond$ to $\boxdot$, i.e., $x \le \Box^{\beta} x$, $\ulcorner x \le \Box^{\eta}_l x, \ldots$ do not hold, since in general $\ulcorner\overline{x} \ne \neg\ulcorner x$.
In sum all theorems given in [18, 14, 17] hold in the generalisation, too. Most of them are already proved by the Galois connection and the Lemmas above. For example, we get the following translation table between [18] and our approach.

Theorems of [18]	**related Lemma**
NL1	4.7.(i)
NL2	4.8.(i)
NL3	4.7.(ii)
NL4	4.7.(ii) and 4.4
NL5	4.7.(iii)
NL6	4.7.(iii)

Another important property is again a cancellative law for neighbours.

Lemma 4.9

$$
\begin{array}{rlcl}
(i) & \Diamond^{\eta}_l \Diamond^{\eta}_r y = \Diamond^{\beta}_r y & \textit{and} & \Diamond^{\eta}_r \Diamond^{\eta}_l y = \Diamond^{\beta}_l y, \\
(ii) & \Diamond^{\beta}_l \Diamond^{\eta}_r y = \Diamond^{\eta}_r y & \textit{and} & \Diamond^{\beta}_r \Diamond^{\eta}_l y = \Diamond^{\eta}_l y, \\
(iii) & \Diamond^{\beta}_l \Diamond^{\beta}_l y = \Diamond^{\beta}_l y & \textit{and} & \Diamond^{\beta}_r \Diamond^{\beta}_r y = \Diamond^{\beta}_r y, \\
(iv) & \Diamond^{\eta}_l \Diamond^{\beta}_l y = \Diamond^{\eta}_l y & \textit{and} & \Diamond^{\eta}_r \Diamond^{\beta}_r y = \Diamond^{\eta}_r y.
\end{array}
$$

Proof. (i) $\Diamond^{\eta}_l \Diamond^{\eta}_r y = \Diamond^{\eta}_l (y^{\urcorner} \cdot \top) = \top \cdot {}^{\ulcorner}(y^{\urcorner} \cdot \top) = \top \cdot y^{\urcorner} = \Diamond^{\beta}_r y$
(ii) $\Diamond^{\beta}_l \Diamond^{\eta}_r y = \Diamond^{\beta}_l (y^{\urcorner} \cdot \top) = {}^{\ulcorner}(y^{\urcorner} \cdot \top) \cdot \top = y^{\urcorner} \cdot \top = \Diamond^{\eta}_r y$
(iii) $\Diamond^{\beta}_l \Diamond^{\beta}_l y = \Diamond^{\beta}_l ({}^{\ulcorner}y \cdot \top) = {}^{\ulcorner}({}^{\ulcorner}y \cdot \top) \cdot \top = {}^{\ulcorner}y \cdot \top = \Diamond^{\beta}_l y$
(iv) $\Diamond^{\eta}_l \Diamond^{\beta}_l y = \Diamond^{\eta}_l ({}^{\ulcorner}y \cdot \top) = \top \cdot {}^{\ulcorner}({}^{\ulcorner}y \cdot \top) = \top \cdot {}^{\ulcorner}y = \Diamond^{\eta}_l y$

□

Lemma 4.9 implies an analogous lemma for boxes via the de Morgan dualities. Hence, for any combination of two boxes $\Box\Box y = \overline{\Diamond\Diamond\overline{y}}$, we have the following corollary.

Corollary 4.10

$$
\begin{array}{rlcl}
(i) & \Box^{\eta}_l \Box^{\eta}_r y = \Box^{\beta}_r y & \textit{and} & \Box^{\eta}_r \Box^{\eta}_l y = \Box^{\beta}_l y, \\
(ii) & \Box^{\beta}_l \Box^{\eta}_r y = \Box^{\eta}_r y & \textit{and} & \Box^{\beta}_r \Box^{\eta}_l y = \Box^{\eta}_l y, \\
(iii) & \Box^{\beta}_l \Box^{\beta}_l y = \Box^{\beta}_l y & \textit{and} & \Box^{\beta}_r \Box^{\beta}_r y = \Box^{\beta}_r y, \\
(iv) & \Box^{\eta}_l \Box^{\beta}_l y = \Box^{\eta}_l y & \textit{and} & \Box^{\eta}_r \Box^{\beta}_r y = \Box^{\eta}_r y.
\end{array}
$$

As a last simplification of NL, we show that Axiom 6 of [17] is now a theorem.

Lemma 4.11 $\Diamond^{\eta}_l \Diamond^{\eta}_r y = \Box^{\eta}_l \Diamond^{\eta}_r y$ *and* $\Diamond^{\eta}_r \Diamond^{\eta}_l y = \Box^{\eta}_r \Diamond^{\eta}_l y$.

Proof. The direction ($\geq$) is immediate by Lemma 4.4, whereas ($\leq$) can be shown as follows:

$\Diamond^{\eta}_l \Diamond^{\eta}_r y \leq \Box^{\eta}_l \Diamond^{\eta}_r y$

$\Leftrightarrow$ $\{\![$ Galois connection $]\!\}$

$\Diamond^{\eta}_r \Diamond^{\eta}_l \Diamond^{\eta}_r y \leq \Diamond^{\eta}_r y$

$\Leftrightarrow$ $\{\![$ Lemma 4.9.(i) $]\!\}$

$\Diamond^{\eta}_r \Diamond^{\beta}_r y \leq \Diamond^{\eta}_r y$

$\Leftrightarrow$ $\{\![$ Lemma 4.9.(iv) $]\!\}$

$\Diamond^{\eta}_r y \leq \Diamond^{\eta}_r y$

$\Leftrightarrow$ true

□

There are many more simplifications and extensions for NL which we do not discuss here. We only want to show a much simpler form of $\Box_r \Box_r \Box_l \Box_l \phi$ (reads "for all intervals: ϕ"). This expression was used in [14, 18] for a deduction theorem and is hard to understand and very unhandy (for example because of its size). In our notation we have to look at $\Box^{\eta}_l \Box^{\eta}_l \Box^{\eta}_r \Box^{\eta}_r \mathbb{I}_{\phi}$. Unfortunately, the following simplification is not valid for all bidomain. In a bidomain semiring the greatest element $\top$ *weakly dominates* predicates iff for all $p \in \mathsf{test}(S)\backslash\{0\}$

$$\ulcorner(\top \cdot (p \cdot \top)\urcorner) \;=\; 1 \quad \text{and} \quad (\ulcorner(\top \cdot p) \cdot \top)\urcorner \;=\; 1 \tag{WTP}$$

For example INT as well as REL satisfy (WTP). Now we can shorten the formulae $\Box^{\eta}_l \Box^{\eta}_l \Box^{\eta}_r \Box^{\eta}_r y$.

Lemma 4.12 *If a bidomain semiring satisfies (WTP) then*

(i) $\Diamond^{\eta}_l \Diamond^{\eta}_l \Diamond^{\eta}_r \Diamond^{\eta}_r y \;=\; \Diamond^{\eta}_r \Diamond^{\eta}_r \Diamond^{\eta}_l \Diamond^{\eta}_l y \;=\; \begin{cases} 0 & \text{if } y = 0 \\ \top & \text{otherwise,} \end{cases}$

(ii) $\Box^{\eta}_l \Box^{\eta}_l \Box^{\eta}_r \Box^{\eta}_r y \;=\; \Diamond^{\eta}_r \Diamond^{\eta}_r \Diamond^{\eta}_l \Diamond^{\eta}_l y \;=\; \begin{cases} \top & \text{if } y = \top \\ 0 & \text{otherwise.} \end{cases}$

Proof. (i) By Lemma 4.9.(i) we get $\Diamond^{\eta}_l \Diamond^{\eta}_l \Diamond^{\eta}_r \Diamond^{\eta}_r y \;=\; \Diamond^{\eta}_l \Diamond^{\beta}_r \Diamond^{\eta}_r y$. Using the explicit form of neighbours we get $\Diamond^{\eta}_l \Diamond^{\beta}_r \Diamond^{\eta}_r y \;=\; \top \cdot \ulcorner(\top \cdot (\overline{y}\urcorner \cdot \top)\urcorner)$. Now we can use (WTP) and get the claim.

(ii) Immediate by $\Box^{\eta}_l \Box^{\eta}_l \Box^{\eta}_r \Box^{\eta}_r y \;=\; \overline{\Diamond^{\eta}_l \Diamond^{\eta}_l \Diamond^{\eta}_r \Diamond^{\eta}_r \overline{y}}$, by $\overline{\top} \;=\; 0$ and (i).

$\square$

Note that it is also possible that a bidomain semiring fulfils only one of the equations of (WTP). Then either $\Diamond^{\eta}_l \Diamond^{\eta}_l \Diamond^{\eta}_r \Diamond^{\eta}_r y$ or $\Diamond^{\eta}_r \Diamond^{\eta}_r \Diamond^{\eta}_l \Diamond^{\eta}_l y$ satisfies Lemma 4.12 and therefore

$$\Diamond^{\eta}_l \Diamond^{\eta}_l \Diamond^{\eta}_r \Diamond^{\eta}_r y \neq \Diamond^{\eta}_r \Diamond^{\eta}_r \Diamond^{\eta}_l \Diamond^{\eta}_l y \ .$$

The last properties we want to discuss are reflecting those situations where $\boxdot$ collapses to 0 and $\dot{\Diamond}$ becomes the greatest element. We call an element x *surjective* if $1 \leq x\urcorner$ and *total* if $1 \leq \ulcorner x$.

Lemma 4.13 *(i) If x is surjective, then*

$$\dot{\Diamond}_r x = \top \qquad \textit{and} \qquad \boxdot_r \overline{x} = 0 \ .$$

(ii) If x is total, then

$$\dot{\Diamond}_l x = \top \qquad \textit{and} \qquad \boxdot_l \overline{x} = 0 \ .$$

The proof is immediate from the explicit form of neighbours and boundaries.

5 Other interpretations of neighbours

In the remainder we have a look at the interpretations of (perfect) neighbours and (perfect) boundaries in different semirings. We will show that the interpretations vary from interval properties already shown by Zhou, Hansen and Roy over reachability in graphs to an application to hybrid systems.

5.1 Neighbours in LAN and REL

The formal languages can be made into a semiring by setting

$$\mathrm{LAN}(\Sigma) \stackrel{\text{def}}{=} (\mathcal{P}(\Sigma^*), \cup, ., \emptyset, \{\varepsilon\}) \ ,$$

where $\mathcal{P}(\Sigma^*)$ denotes the set of languages over some finite alphabet Σ, $\cup$ denotes set union and $L_1.L_2 = \{vw \,|\, v \in L_1, w \in L_2\}$, where vw is the concatenation of v and w. Furthermore $\emptyset$ denotes the empty language and ε the empty word. Since $\mathsf{test}(\mathrm{LAN})$ is discrete, i.e., $\mathsf{test}(\mathrm{LAN}) = \{\emptyset, \{\varepsilon\}\}$, we have

$$\ulcorner L = L^{\urcorner} = \begin{cases} \{\varepsilon\} \text{ if } L \neq \emptyset \\ \emptyset \quad \text{otherwise} \end{cases}$$

Thus we have, as in all bidomain semirings with discrete test,

$$\Diamond L = \begin{cases} 0 \ \text{ if } L = \emptyset \\ \top \text{ otherwise,} \end{cases}$$

$$\boxdot L = \begin{cases} \top \text{ if } L = \top \\ 0 \ \text{ otherwise.} \end{cases}$$

That is why all diamonds ($\Diamond^{\eta}_l$, $\Diamond^{\eta}_r$, $\Diamond^{\beta}_l$, $\Diamond^{\beta}_r$) as well as all boxes collapse to one sort of diamonds and boxes, respectively.

In REL the situation is also easy. But before calculating the neighbours in REL, we recapitulate the definition of the semiring of binary relations.

Consider an arbitrary set M and the structure

$$\mathrm{REL}(M) \stackrel{\text{def}}{=} (\mathcal{P}(M \times M), \cup, \circ, \emptyset, \Delta) \ ,$$

where $\cup$ denotes again set union, $\circ$ denotes relation composition, $\emptyset$ is the empty relation and Δ denotes the identity relation $\{(m,m) \,|\, m \in M\}$. Then $\mathrm{REL}(M)$ (or for short just REL) forms an idempotent semiring where the natural order coincides with the subset relation and $\top = \{(x,y) \,|\, x, y \in M\}$.
REL can be extended to a bidomain semiring by defining the test set as

$$\mathsf{test}(\mathrm{REL}) \stackrel{\text{def}}{=} \{R \,|\, R \subseteq \Delta\}$$

and the domain function, similarly as in INT, as

$$\ulcorner R = \{(p,p) \,|\, (p,x) \in R\} \ .$$

Analogously we have

$$R^{\urcorner} = \{(p,p) \,|\, (x,p) \in R\} \ .$$

For an element $P \in \mathsf{test}(\mathrm{REL})$ $P \circ \top$ restricts the first component relations, i.e., $P \circ \top = \{(p,x) \,|\, (p,p) \in P,\ x \in M\}$, whereas $\top \circ P$ restrict the second component, the range. First, we show this circumstance with some examples.

$$\Diamond^{\eta}_r R = R^{\urcorner} \circ \top = \{(x,y) \,|\, \exists w : (w,x) \in R,\ y \in M\}$$

is the set of all pairs (x, y) for which there is a relation $r \in R$ where the composition between r and (x, y) is defined. So, $\langle\eta\rangle_r R$ is the set of pairs that can be composed to R from the right, whereas $\langle\eta\rangle_l$ contains all pairs that can be composed to R from the left. For $[\eta]_r R$ we calculate

$$
\begin{aligned}
[\eta]_r R &= \neg\, (\overline{R})^{\urcorner} \circ \top \\
&= \{(x, y) \,|\, (x, x) \in \neg\, (\overline{R})^{\urcorner},\, y \in M\} \\
&= \{(x, y) \,|\, (x, x) \notin\, (\overline{R})^{\urcorner},\, y \in M\} \\
&= \{(x, y) \,|\, \forall\; w : (w, x) \notin \overline{R},\, y \in M\} \\
&= \{(x, y) \,|\, \forall\; w : (w, x) \in R,\, y \in M\}\ .
\end{aligned}
$$

Hence, $[\eta]_r R$ is the set of all pairs, whose "predecessors" (elements which can be composed from the left) are all elements of R. As already mentioned, REL satisfies (WTP). Thus, we have the cancelative laws of Lemma 4.12.

5.2 Reachability – Neighbours in PAT

Following [4] we can describe graphs as elements of an idempotent bidomain semiring. Consider a set of vertices Σ. Then subsets of Σ^* can be viewed as sets of possible graph paths. The partial operation of *join* or *fusion product* of elements of Σ^* is defined as

$$
\begin{gathered}
\varepsilon \bowtie \varepsilon \\
\varepsilon \bowtie (y.t) \text{ is undefined} \\
(s.x) \bowtie \varepsilon \text{ is undefined} \\
(s.x) \bowtie (y.t) \;=\; \begin{cases} s.x.t & \text{if } x \,=\, y \\ \text{undefined} & \text{otherwise} \end{cases}
\end{gathered}
$$

for all $s, t \in \Sigma^*$ and $x, y \in \Sigma$. It describes the "gluing" of paths at a common point. This operation is extended to subsets of Σ^* by

$$S \bowtie T \;=\; \{s \bowtie t | s \in S,\, t \in T,\, s \bowtie t \text{ is defined}\}\ .$$

Then $\mathrm{PAT}(\Sigma) \;=\; (\mathcal{P}(\Sigma^*), \cup, \bowtie, \emptyset, \Sigma \cup \{\varepsilon\})$ forms an idempotent semiring which can be extended to a bidomain semiring, where $^{\ulcorner}$ describes the starting points of the paths, i.e.,

$$^{\ulcorner}S \;=\; \{x | (x.s) \in S\} \cup \begin{cases} \varepsilon & \text{if } \varepsilon \in S \\ \emptyset & \text{otherwise}\ . \end{cases}$$

Analogously, $^{\urcorner}$ characterises sets of endpoints. $\langle\eta\rangle_r S$ is the set of all vertex sequences that start in an endpoint of S. In other words $\langle\eta\rangle_r S$ describes all paths which are reachable through S.

Similarly to the calculations in REL we get $\boxed{\neg}_r$ by

$$
\begin{aligned}
\boxed{\neg}_r S &= \neg\,(\overline{S})^{\urcorner} \bowtie \top \\
&= \{x.t \,|\, x \in \neg\,(\overline{S})^{\urcorner},\ x.t \in \top\} \\
&= \{x.t \,|\, x \notin (\overline{S})^{\urcorner},\ x.t \in \Sigma^*\} \\
&= \{x.t \,|\, \forall\ s \in \Sigma^*\ :\ s.x \notin \overline{S},\ t \in \Sigma^*\} \\
&= \{x.t \,|\, \forall\ s \in \Sigma^*\ :\ s.x \in S,\ t \in \Sigma^*\}
\end{aligned}
$$

Hence $\boxed{\neg}_r S$ is the set of those paths which can only be reached from S, not from $\overline{S}$. PAT is an example where (WTP) doesn't hold. Hence, we do not have the cancelative laws of Lemma 4.12. It is also a counterexample for a bidomain semiring where $\boxed{\neg}_l \boxed{\neg}_l \boxed{\neg}_r \boxed{\neg}_r y \neq \boxed{\neg}_r \boxed{\neg}_r \boxed{\neg}_l \boxed{\neg}_l y$.

5.3 Neighbours in PRO – Applications in Hybrid Systems

In [11] we introduced an algebra of *processes*. Processes are sets of trajectories and are very useful for describing hybrid systems in an algebraic way. In the paper we use finite trajectories as well as infinite ones. Admitting the latter ones entails that we have no idempotent semiring anymore. The situation changes when we restrict ourself to finite trajectories.
Again we briefly repeat the definitions. A *trajectory* is a pair (d, g), where $d \in \mathbb{T}\mathrm{ime}$ and $g : [0, d] \to V$, where V is a set of *values*. Here, we only have intervals with finite length and therefore we have only finite trajectories. We define composition of trajectories (d_1, g_1) and (d_2, g_2) as

$$
(d_1, g_1) \cdot (d_2, g_2) \stackrel{\text{def}}{=} \begin{cases} (d_1 + d_2, g) \text{ if } g_1(d_1) = g_2(0) \\ \text{undefined} \quad \text{otherwise} \end{cases}
$$

with $g(x) = g_1(x)$ for all $x \in [0, d_1]$ and $g(x + d_1) = g_2(x)$ for all $x \in [0, d_2]$. Composition is lifted to processes pointwise, i.e., for processes A, B we have $A \cdot B \stackrel{\text{def}}{=} \{a \cdot b \,|\, a \in A,\ b \in B,\ a \cdot b \text{ is defined}\}$. The set of all trajectories is denoted by TRA and we denote for a value $v \in V$ the corresponding zero-length trajectory by $\underline{v} \stackrel{\text{def}}{=} (0, g)$, where $g(0) = v$. Then the structure

$$
\mathrm{PRO} \stackrel{\text{def}}{=} (\mathcal{P}(\mathrm{TRA}), \cup, \emptyset, \cdot, I, {}^{\ulcorner}, {}^{\urcorner})
$$

forms a bidomain quantale with $\mathsf{test}(\mathrm{PRO}) = \mathcal{P}(\{\underline{v} \,|\, v \in V\})$, ${}^{\ulcorner}A = \{\underline{g(0)} \,|\, (d, g) \in A\}$ and $A^{\urcorner} = \{\underline{g(d)} \,|\, (d, g) \in A\}$. Since trajectories include as one component intervals, the behaviour of (perfect) neighbours and (perfect) boundaries are as in INT. On the other hand trajectories contain functions, hence (perfect) neighbours and boundaries are as in PAT. Together they are just a combination of both. In contrast to PAT, (WTP) holds in PRO. Since we have presented the neighbours and boundaries of INT and PAT we do not want to discuss neighbours in PRO. However they are very useful in calculations for hybrid systems.

6 Adding $*$ and ω

Following [3] every quantale can be extended to a Kleene algebra by the definition of $a^* \stackrel{\text{def}}{=} \mu x.a \cdot x + 1$. If the quantale is even a complete distributive lattice then it can be extended to an ω-algebra by setting $a^\omega \stackrel{\text{def}}{=} \nu x.a \cdot x$. Hence INT as well as PRO form Kleene and ω-algebras. In the remainder we want to discuss the effects of $*$ and ω on the neighbour modalities. But first we want to recapitulate the basic definitions.

A Kleene algebra is a pair $(S,^*)$, where S is an idempotent semiring and * satisfies the following unfold and induction laws.

$$1 + a \cdot a^* \ \leq \ a^*, \tag{1}$$
$$1 + a^* \cdot a \ \leq \ a^*, \tag{2}$$
$$b + a \cdot c \leq c \ \Rightarrow \ a^* \cdot b \leq c, \tag{3}$$
$$b + c \cdot a \leq c \ \Rightarrow \ b \cdot a^* \leq c. \tag{4}$$

An ω-algebra is a pair $(S, {}^\omega)$, where S is a Kleene algebra and $^\omega$ satisfies

$$a^\omega \ \leq \ a \cdot a^\omega, \tag{5}$$
$$c \leq b + a \cdot c \ \Rightarrow \ c \leq a^\omega + a^* \cdot b. \tag{6}$$

$*$ characterises finite iteration and ω infinite iteration. So, for example, one can describe loops and other repeating procedures with these operators. A Kleene algebra (ω-algebra) is called *bidomain* iff the underlying semiring is a bidomain semiring. If we set $a^+ \stackrel{\text{def}}{=} a \cdot a^*$, we get useful properties for neighbours and boundaries.

Lemma 6.1 *(i)* $\diamondsuit y^* \ = \ \top$.
(ii) $x^* \leq \diamondsuit_l y \ \Leftrightarrow \ 1 \leq \ulcorner y$,
$x^* \leq \diamondsuit_r y \ \Leftrightarrow \ 1 \leq y\urcorner$.
(iii) $\diamondsuit y^+ \ = \ \diamondsuit y$
(iv) $x^+ \leq \diamondsuit y \ \Leftrightarrow \ x \leq \diamondsuit y$

Proof. By definition, $\ulcorner(x^*) \ = \ (x^*)\urcorner \ = \ 1$ and $\ulcorner(x^+) \ = \ \ulcorner x, \ (x^+)\urcorner \ = \ x\urcorner$. $\square$

In ω-algebras the situation is much more complicated, since the domain/co-domain operators do not behave symmetrically. Hence we first have a look at ω and domain.

Lemma 6.2 *(i)* $\ulcorner(a^\omega) \leq \ulcorner a$.
If a is dense, i.e., $a \leq a \cdot a$, we have $\ulcorner(a^\omega) \ = \ \ulcorner a$.
(ii) If a is dense, we have $a\urcorner \ \leq \ (a^\omega)\urcorner$

Proof. (i) $\ulcorner(a^\omega) \ = \ \ulcorner(a \cdot a^\omega) \leq \ulcorner a$.
By (6) we get $a \leq a \cdot a \ \Rightarrow \ a \leq a^\omega$ and the claim follows by isotonicity.
(ii) Again by (6) and isotonicity.

$\square$

Now we briefly discuss the interaction between the ω-operator and neighbours or boundaries

Lemma 6.3 *(i)* $x^\omega \leq \Diamond^{\eta}_r y \Rightarrow x \leq \Diamond^{\eta}_r y$,
$x^\omega \leq \Diamond^{\beta}_l y \Rightarrow x \leq \Diamond^{\beta}_l y$.

(ii) If x is dense, we have
$x \leq \Diamond^{\cdot} y \Rightarrow x^\omega \leq \Diamond^{\cdot} y$

Proof. (i) By definition, Lemma 6.2 and definition again.

$$x^\omega \leq \Diamond^{\eta}_r y \Leftrightarrow \ulcorner(x^\omega) \leq y\urcorner \Leftarrow \ulcorner x \leq y\urcorner \Leftrightarrow x \leq \Diamond^{\eta}_r y$$

(ii) Similar to (i).

□

Unfortunately we do not have similar properties for perfect neighbours and perfect boundaries.

7 Conclusion and Outlook

In the paper we started with the Neighbourhood Logic developed by Zhou and Hansen. We showed how to embed NL into the theory of semirings. With the help of the embedding we showed that at least two axioms can be dropped in the definition of NL and that neighbours can be expressed in a much more general framework. Therefore we presented neighbours and boundaries in bidomain semirings and presented important Galois connections. Then we discussed neighbours and boundaries in many different models. E.g., we showed properties of reachability in the path algebra and a useful interpretation for hybrid systems. At the end we showed how the neighbours and boundaries interact with finite and infinite iteration in the algebraic structures of Kleene algebra and ω-algebra.

Möller developed the theory of lazy semirings and we presented an algebra for hybrid systems in [11]. This model handles finite trajectories as well as infinite trajectories. Thus we want to adapt and, if necessary, modify the neighbours and boundaries for the case of lazy semirings. Then we have a further application for NL in a theory where we can express unlimited processes. This second extension step should also show a connection between [1] and NL.

Acknowledgements

I am very grateful to Bernhard Möller for his valuable comments and remarks, especially concerning the explicite formulation of the neighbours. Useful comments were also provided by Kim Solin.

References

1. Zhou Chaochen, Dang Van Hung, and Li Xiaoshan. A duration calculus with infinite intervals. In *Fundamentals of Computation Theory*, pages 16–41, 1995.
2. J. H. Conway. *Regular Algebra and Finite Machines*. Chapman and Hall, 1971.
3. J. Desharnais and B. Möller. Characterizing determinacy in kleene algebra. *Special Issue on Relational Methods in Computer Science, Information Sciences — An International Journal*, 139:253–273, 2001.
4. J. Desharnais, B. Möller, and G. Struth. Kleene algebra with domain. *ACM Transaction on Computational Logic*, 2004. Preliminary version: Universität Augsburg, Institut für Informatik, Report No. 2003-07, June 2003.
5. Bruno Dutertre. Complete proof systems for first order interval temporal logic. In *LICS*, pages 36–43, 1995.
6. J. S. Golan. *The Theory of Semirings with Applications in Mathematics and Theoretical Computer Science*. Number ISBN 0-582-07855-5. Longman, 1992.
7. Joseph Y. Halpern and Y. Shoham. A propositional modal logic of time intervals. In *Proceedings 1st Annual IEEE Symp. on Logic in Computer Science, LICS'86, Cambridge, MA, USA, 16–18 June 1986*, pages 279–292. IEEE Computer Society Press, Washington, DC, 1986.
8. J.Y. Halpern, Moszkowski B., and Manna Z. A Hardware Semantics based on Temporal Intervals. In Diaz J., editor, *ICALP'83*, number 154 in Lecture Notes in Computer Science, pages 278–291. Springer, Berlin, 1983.
9. U. Hebisch and H. J. Weinert. *Semirings - Algebraic Theory and Applications in Computer Science*. World Scientific, Singapur, January 1998.
10. P. Höfner. From Sequential Algebra to Kleene Algebra: Interval Modalities and Duration Calculus. Technical Report 2005-05, Universität Augsburg, 2005.
11. P. Höfner and B. Möller. Towards an Algebra of Hybrid Systems. *Proceedings of the 8th International Conference on Relational Methods in Computer Science (RelMiCS 8)*, February 2005.
12. D. Kozen. Kleene algebra with tests. *Transactions on Programming Languages and Systems*, 19(3):427–443, May 1997.
13. B. Möller. Residuals and Detachments. Technical Report 2005-20, Universität Augsburg.
14. S. Roy and C. Zhou. Notes on Neighbourhood Logic. Technical Report 97, The United Nations University UNU/IIST, February 1997.
15. Y. Venema. A modal logic for chopping intervals. *J. Log. Comput.*, 1(4):453–476, 1991.
16. C. Zhou and M.R. Hansen. An Adequate First Order Interval Logic. Technical Report 91, The United Nations University UNU/IIST, December 1996.
17. C. Zhou and M.R. Hansen. An Adequate First Order Interval Logic. *Lecture Notes in Computer Science*, 1536:584–608, 1998.
18. C. Zhou and M.R. Hansen. *Duration Calculus – A Formal Approach to Real-Time Systems*. Springer, Monographs in Theoretical Computer Science, February 2004. ISBN:3540408231.